

$D_4$  MODULAR FORMS

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ABSTRACT. In this paper, we study modular forms on two simply connected groups of type  $D_4$  over  $\mathbb{Q}$ . One group,  $\mathbf{G}_s$ , is a globally split group of type  $D_4$ , viewed as the group of isotopies of the split rational octonions. The other,  $\mathbf{G}_c$ , is the isotopy group of the rational (non-split) octonions. We study automorphic forms on  $\mathbf{G}_s$  in analogy to the work of Gross, Gan, and Savin on  $G_2$ ; namely we study automorphic forms whose component at infinity corresponds to a quaternionic discrete series representation. We study automorphic forms on  $\mathbf{G}_c$  using Gross's formalism of "algebraic modular forms". Finally, we follow work of Gan, Savin, Gross, Rallis, and others, to study an exceptional theta correspondence connecting modular forms on  $\mathbf{G}_c$  and  $\mathbf{G}_s$ . This can be thought of as an octonionic generalization of the Jacquet-Langlands correspondence.

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## INTRODUCTION

The aim of this paper is to study modular forms on two absolutely simple, simply-connected reductive groups over  $\mathbb{Q}$  of type  $D_4$ . Originally, I chose to write a paper on this subject because I was interested in the work of Gross, Savin, and Gan (especially in [15]) on  $G_2$ , and I thought I could gain a deep understanding of their methods by writing a paper on the similar phenomena in  $D_4$ . This paper was originally meant to be an extended (and hopefully flattering) exercise in imitation. As a result, the techniques used are for the most part not new, and may be found scattered among the papers on  $G_2$ .

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After writing this paper, I believe that though the techniques may not be new, the results may have independent interest. Specifically, I believe that the following results in this paper are new, or at least differ from those for  $G_2$ :

- The Fourier coefficients of modular forms on the split group of type  $D_4$  are indexed by triples in the ideal class group of imaginary quadratic fields, whose product is the principal class. Thus, as Siegel’s modular forms of degree 2 encode information about ideal class groups of imaginary quadratic fields,  $D_4$  modular forms may encode information on the fine structure of these ideal class groups.
- The geometry of generalized flag varieties for the split  $D_4$  is expressed in terms of octonions. While essentially contained in the work of J. Tits [35], I hope that the reader finds the simplicity of this approach appealing.
- The dual pair correspondence between the split  $D_4$  and the  $D_4$  which is split at every finite place, and compact at the real place, should be thought of as an octonionic generalization of the Jacquet-Langlands correspondence. This is a unique phenomenon for  $D_4$ , and the existence of this correspondence (as suggested to me by B. Gross) is a primary motivation for this paper.
- One may construct modular forms on the non-split  $D_4$  through the invariant polynomials for the  $E_8$  Weyl group.

There are many further directions in the study of  $D_4$  modular forms. The action of Hecke operators on the Fourier coefficients of  $D_4$  modular forms may be computed as in [15] using the relative Satake transform. More precise information about the  $D_4$ - $D_4$  dual pair correspondence would be nice, especially relating to values of L-functions. One may also study modular forms on outer forms of  $D_4$  associated to totally real cubic étale algebras over  $\mathbb{Q}$ , and we hope to return to this topic later.

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## BACKGROUND AND NOTATION

$\mathbb{A}$  will always denote the adeles of the field of rational numbers.  $\hat{\mathbb{Z}}$  denotes the profinite completion of  $\mathbb{Z}$ , and is viewed as a  $\mathbb{Z}$ -algebra, and a subalgebra of  $\mathbb{A}$ .  $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$  will denote the finite adeles.

We use different typefaces for schemes, algebraic varieties, and the points of algebraic varieties. If  $R$  is a ring, we use an underlined letter, such as  $\underline{\mathbf{S}}$  to denote a scheme over  $R$ . If  $k$  is a field, we use a boldface letter, such as  $\mathbf{S}$  to denote an algebraic variety over  $k$ . Finally, if the field  $k$  is fixed, we write  $S = \mathbf{S}(k)$  for the set of  $k$ -points of  $\mathbf{S}$ ; if  $k$  is a field with a natural topology, such as  $\mathbb{Q}_p$  or  $\mathbb{R}$ , we endow  $S$  with the resulting topology when possible.

If  $G$  and  $G'$  are groups, and  $V, V'$  are representations of  $G, G'$  respectively, then we write  $V \boxtimes V'$  for the “external tensor product” representation of  $G \times G'$ . On the other hand, if  $V, V'$  are representations of a single group  $G$ , we write  $V \otimes V'$  for the usual tensor product of representations, i.e.,  $V \otimes V'$  is a representation of  $G$ , while  $V \boxtimes V'$  is a representation of  $G \times G$ .

If  $G$  is a group, and  $V$  is a  $G$ -module, then we write  $V^G$  for the subspace of  $G$ -fixed elements of  $V$ . We write  $V_G$  for the maximal  $G$ -invariant quotient of  $V$ :  $V_G = V/\langle gv - v \rangle$ .

We frequently use basic facts about affine algebraic groups, as discussed in Waterhouse [38]. By an algebraic group  $\underline{G}$  over a ring  $R$ , we always mean an affine algebraic group. For algebraic groups over a field  $k$ , we remove the underline, and write  $\mathbf{G}$ . If  $R$  is a  $k$ -algebra, we let  $\mathbf{G}(R)$  denote the  $R$ -points of  $G$ . If  $\underline{G}$  is an affine group scheme over  $\mathbb{Z}$ , the base change to  $\mathbb{Q}$  will be implicitly denoted by removing the underline:  $\mathbf{G} = \underline{G} \otimes_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Q})$ .  $\mu_n$  will denote the group scheme of  $n^{\text{th}}$  roots of unity.  $\mathbb{G}_a$  will denote the additive group scheme, and  $\mathbb{G}_m$  will denote the multiplicative group scheme.

If  $\mathbf{G}$  is a reductive algebraic group over  $\mathbb{Q}$ , we consider three types of automorphic objects for  $\mathbf{G}$ .

First, we have the space  $\mathcal{A} = \mathcal{A}(\mathbf{G})$  of *automorphic forms* on  $\mathbf{G}$ , as discussed in the article of Borel [5].  $\mathcal{A}$  is defined to be the space of smooth functions  $f$  on  $\mathbf{G}(\mathbb{A})$  such that:

- $f$  is left-invariant under  $\mathbf{G}(\mathbb{Q})$ .
- $f$  is right-invariant under some open compact subgroup of  $\mathbf{G}(\hat{\mathbb{Q}})$ .
- $f$  is annihilated by an ideal  $J$  of finite codimension in the center of the universal enveloping algebra of the complexified Lie algebra  $\mathfrak{g} \otimes \mathbb{C}$  of  $\mathbf{G}$ .
- $f$  is of uniform moderate growth on  $\mathbf{G}(\mathbb{R})$ .

As in the article [15], and in contrast to Borel [5], we do not assume our automorphic forms  $f$  to be  $K_\infty$ -finite for a maximal compact subgroup  $K_\infty$  of  $G(\mathbb{R})$ .

Let  $\mathcal{A}^0 = \mathcal{A}^0(\mathbf{G})$  denote the cuspidal subspace. Both  $\mathcal{A}$  and its subspace of cusp forms admit actions of  $\mathbf{G}(\mathbb{A})$  by right translation.

Second, we have the set of *automorphic representations* of  $\mathbf{G}$ . We define an *automorphic representation* of  $\mathbf{G}$  to be a pair  $(\pi, \rho)$ , where  $\pi$  is an irreducible admissible representation of  $\mathbf{G}(\mathbb{A})$ , and  $\rho$  (the realization) is a  $\mathbf{G}(\mathbb{A})$ -intertwining homomorphism from  $\pi$  into the space  $\mathcal{A}(\mathbf{G})$  of automorphic forms. We say that  $(\pi, \rho)$  is cuspidal if the image of  $\rho$  lies in the subspace of cuspidal automorphic forms. Again, the notion of automorphic representation depends only on the variety  $\mathbf{G}$  and not on the integral structure.

Third, we have the notion of a *modular form* for  $\mathbf{G}$ . We define a *weight* to be an irreducible smooth representation of  $\mathbf{G}(\mathbb{R})$  on a complex Fréchet space  $W$ . A *level* will be an open compact subgroup  $\hat{K}$  of  $\underline{G}(\hat{\mathbb{Q}})$ ; if  $0 < N$  is an integer, and  $\mathbf{G}$  comes from a group scheme  $\underline{G}$  over  $\mathbb{Z}$  with good reduction everywhere, we associate to  $N$  the open compact subgroup

$$\hat{K}(N) = \ker \left( \underline{G}(\hat{\mathbb{Z}}) \rightarrow \underline{G}(\mathbb{Z}/N\mathbb{Z}) \right),$$

and refer to  $N$  as a level by abuse of notation.

The space of modular forms of weight  $W$  and level  $\hat{K}$  is defined to be the space of  $\mathbf{G}(\mathbb{R}) \times \hat{K}$  intertwining homomorphisms from  $W \boxtimes \mathbb{C}$  to the space  $\mathcal{A}(\mathbf{G})$  of automorphic forms on  $\mathbf{G}$ . The space of cusp forms is defined to be the subspace of homomorphisms whose image lies in the space of cuspidal automorphic forms.

Associated to the open compact subgroup  $\hat{K}$ , we may form the global Hecke algebra  $\mathcal{H}(\hat{K})$  consisting of compactly supported, bi- $\hat{K}$ -invariant functions on  $\mathbf{G}(\hat{\mathbb{Q}})$ . The space of modular forms of weight  $W$  and level  $\hat{K}$  clearly admits an action of

this Hecke algebra. Irreducible Hecke submodules of the space of modular forms yield automorphic representations in the usual manner.

Suppose that  $\mathbf{G}$  is an algebraic group over  $\mathbb{Q}$ , and  $\mathbf{G}(\mathbb{Q})$  is discrete in  $\mathbf{G}(\mathbb{A})$  with finite co-volume. If  $f$  is a (measurable) function on  $\mathbf{G}(\mathbb{A})$  that is left-invariant under  $\mathbf{G}(\mathbb{Q})$ , we use the shorthand:

$$\oint_{\mathbf{G}} f(g) dg = \int_{\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A})} f(g) dg,$$

for the integral with respect to Tamagawa measure. Note that for unipotent groups  $\mathbf{U}$ , Tamagawa measure is normalized so that the compact quotient  $\mathbf{U}(\mathbb{Q}) \backslash \mathbf{U}(\mathbb{A})$  has volume 1.

A few such integrals will arise repeatedly, and we mention them here. If  $\mathbf{G}$  is a reductive group over  $\mathbb{Q}$ ,  $\mathbf{U}$  is a unipotent  $\mathbb{Q}$ -subgroup of  $\mathbf{G}$ , and  $f$  is an automorphic form on  $\mathbf{G}$ , then we define the  $\mathbf{U}$ -constant term of  $f$  by the integral:

$$f_{\mathbf{U}}(g) = \oint_{\mathbf{U}} f(ug) du.$$

More generally, if  $\phi$  is a character of  $\mathbf{U}(\mathbb{A})$  which is trivial on  $\mathbf{U}(\mathbb{Q})$ , we define the  $\phi$ -coefficient of  $f$  to be:

$$f_{\phi}(g) = \oint_{\mathbf{U}} f(ug) \overline{\phi(u)} du.$$

If  $\mathbf{U}$  is abelian unipotent, then the characters of  $\mathbf{U}(\mathbb{A})$  trivial on  $\mathbf{U}(\mathbb{Q})$  can be identified with  $\mathbf{U}(\mathbb{Q})$ , and the Fourier expansion of  $f$  reads:

$$f(g) = \sum_{u \in \mathbf{U}(\mathbb{Q})} f_{\phi_u}(g).$$

## 1. STRUCTURE THEORY

It is possible to construct a split simply connected group scheme of type  $D_4$  over  $\mathbb{Z}$  by the methods of Chevalley [7] using only the root system. However, we prefer to use a construction that is more special to  $D_4$ , and which incorporates the most interesting phenomena (triality and octonionic structure) from the start. From the paper of Gross [18], recall that if  $\mathbf{G}$  is a connected reductive algebraic group over  $\mathbb{Q}$ , then we say that  $\underline{\mathbf{G}}$  is a model for  $G$  over  $\mathbb{Z}$  if  $\underline{\mathbf{G}}$  is a smooth affine group scheme over  $\mathbb{Z}$  with general fibre  $\mathbf{G}$ , and with good reduction *everywhere*. In [18], Gross gives criteria for the existence of models over  $\mathbb{Z}$ , and enumerates or classifies these models in many cases. Specifically,  $\mathbf{G}$  admits a model over  $\mathbb{Z}$  if and only if  $\mathbf{G}$  is split over  $\mathbb{Q}_p$  for *every* prime number  $p$ . From [18] it follows that:

- There is a model  $\underline{\mathbf{G}}_s$  of the split simply connected simple group  $\mathbf{G}_s/\mathbb{Q}$  of type  $D_4$ . At the real place, it satisfies  $\mathbf{G}_s(\mathbb{R}) \cong Spin_{4,4}(\mathbb{R})$ .
- There is a model  $\underline{\mathbf{G}}_c$  of the simply connected group  $\mathbf{G}_c/\mathbb{Q}$  of type  $D_4$  which is split at every finite prime  $p$ , and which is anisotropic (compact) at the real place,  $\mathbf{G}_c(\mathbb{R}) \cong Spin_8(\mathbb{R})$ .
- There is a model  $\underline{\mathbf{E}}_{8,4}$  of the simply connected group of type  $E_8$  which is split at every finite prime  $p$ , and which is the quaternionic form of real rank 4 at the real place.
- There is a model  $\underline{\mathbf{E}}_{7,7}$  of the simply connected group which is split of type  $E_7$  at every finite prime  $p$ , and at the real place as well.

- There is a model  $\underline{\mathbf{E}}_{7,3}$  of the simply connected group which is split of type  $E_7$  at every finite prime  $p$ , and has real rank 3 at the real place.

These groups are described briefly in [18], and the groups of type  $D_n$  are more explicitly constructed in a paper of Goldstine [17]. All of the above integral models are unique in their genus. Loke explores the dual reductive pair  $Spin_{4,4}(\mathbb{R}) \times_{\mu_2 \times \mu_2} Spin_8(\mathbb{R})$  in  $E_{8,4}$  over the reals in [30], and in [16], Ginzburg, Jiang, and Rallis study the dual pair  $\mathbf{G}_s \times (\mathbf{SL}_2^3)$  in  $\mathbf{E}_{7,7}$ . Gross and Savin study dual pairs in  $\mathbf{E}_{7,3}$  in [20]. In this first section we study the structure theory of the groups  $\underline{\mathbf{G}}_s, \underline{\mathbf{G}}_c, \underline{\mathbf{E}}_{8,4}, \underline{\mathbf{E}}_{7,7}, \underline{\mathbf{E}}_{7,3}$  in a way which makes these dual pairs more transparent.

**1.1. Octonions.** The connection between exceptional groups, triality, and the octonions can be found in detail in the recent exposition of Baez [2]. The structure of the octonions over  $\mathbb{Z}$  was first correctly understood by Coxeter in [9]. An excellent recent exposition on octonions, especially over  $\mathbb{Z}$ , is provided by the book of Conway and Smith [8]. The structure of the integral split octonions, as well as connections to exceptional groups can be found in the paper of Krutelevich [29]. First, we explicitly describe the groups  $\underline{\mathbf{G}}_s, \underline{\mathbf{G}}_c$ . We continue using the subscripts  $c$  or  $s$  to remind the reader when an object is associated to a group which is compact or split at infinity respectively.

Begin by letting  $\mathbb{H}_c$  denote the division algebra of Hamilton's quaternions,  $\mathbb{H}_c = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ , with  $ij = -ji = k$ , and  $i^2 = j^2 = k^2 = -1$ . The main involution on the quaternions is given by:

$$\overline{(a + bi + cj + dk)} = a - bi - cj - dk.$$

Let  $\mathbb{H}_s$  denote the “split quaternions”, i.e., the algebra of 2 by 2 matrices with real entries. The main involution on the matrix algebra is given by:

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Let  $Y_s$  denote the maximal order in  $\mathbb{H}_s$  consisting of 2 by 2 matrices with integer entries. Let  $Y_c$  denote the maximal (Eichler) order in  $\mathbb{H}_c$  with  $\mathbb{Z}$ -basis:

$$1, \frac{1}{2}(1 + i + j + k), \frac{1}{2}(1 + i + j - k), \frac{1}{2}(1 - i + j + k).$$

The real numbers  $\mathbb{R}$  can be identified with the center of both  $\mathbb{H}_c$  and  $\mathbb{H}_s$ , and both  $\mathbb{H}_c$  and  $\mathbb{H}_s$  split naturally into the direct sum of  $\mathbb{R}$  and  $Im(\mathbb{H}_c)$ ,  $Im(\mathbb{H}_s)$  respectively. The space  $Im(\mathbb{H}_s)$  is just the set of trace zero matrices, which can then be identified with the Lie algebra of  $SL_2(\mathbb{R})$ .

We frequently use the “wildcard” notation where  $\bullet$  may stand for  $s$  or  $c$ . We may apply the “Cayley-Dickson process” to  $\mathbb{H}_\bullet$  to get two eight-dimensional alternative normed algebras with involution over  $\mathbb{R}$ . We let  $\mathbb{O}_\bullet = \mathbb{H}_\bullet \oplus \mathbb{H}_\bullet$ , with multiplication law:

$$(u, v) \cdot (z, w) = (uz - \bar{w}v, wu + v\bar{z}).$$

We define the main involution by:

$$\overline{(u, v)} = (\bar{u}, -v).$$

This yields the trace and norm:

$$\begin{aligned} Tr(u, v) &= (u, v) + \overline{(u, v)} = (u + \bar{u}), \\ N(u, v) &= (u, v) \cdot \overline{(u, v)}. \end{aligned}$$

The norm on  $\mathbb{O}_c$  and  $\mathbb{O}_s$  gives a quadratic form on an 8-dimensional real vector space of signature  $(8, 0)$  and  $(4, 4)$  respectively. We call the elements of  $\mathbb{O}_c$  and  $\mathbb{O}_s$  octonions and split octonions respectively. In  $\mathbb{O}_s$ , let  $\Omega_s$  be the set of pairs  $(u, v)$  with  $u$  and  $v$  matrices with integer coefficients. Then  $\Omega_s$  is a maximal order in  $\mathbb{O}_s$ . With the symmetric integer-valued bilinear form  $\langle \alpha, \beta \rangle = \text{Tr}(\bar{\alpha}\beta)$ ,  $\Omega_s$  is a globally split lattice. In  $\mathbb{O}_c$ , we let  $\Omega_c$  denote Coxeter's ring of integral octonions from [9]. Though harder to explicitly describe,  $\Omega_c$ , endowed with the symmetric integer-valued bilinear form  $\langle \alpha, \beta \rangle = \text{Tr}(\bar{\alpha}\beta)$ , is isomorphic to the root lattice  $E_8$  (after scaling). We can identify  $\mathbb{H}_\bullet$  as the subalgebra of  $\mathbb{O}_\bullet$  consisting of pairs  $(u, 0)$ . In this way,  $Y_\bullet$  is a subring of  $\Omega_\bullet$  as well.

Though multiplication in the algebras  $\mathbb{O}_\bullet$  is not associative, for  $\alpha, \beta, \gamma \in \mathbb{O}_\bullet$  the real number

$$\text{Tr}(\alpha\beta\gamma) = \text{Tr}(\alpha \cdot (\beta\gamma)) = \text{Tr}((\alpha\beta) \cdot \gamma)$$

is well defined. If moreover,  $\alpha, \beta, \gamma \in \Omega_\bullet$ , then  $\text{Tr}(\alpha\beta\gamma) \in \mathbb{Z}$ , giving trilinear forms on  $\Omega_\bullet$ .

**1.2. Integral models.** We consider  $(\Omega_\bullet, N)$  as an orthogonal  $\mathbb{Z}$ -module, from which we get an sequence of group schemes over  $\mathbb{Z}$ :

$$1 \rightarrow \mu_2 \rightarrow \underline{\text{Spin}}(\Omega_\bullet, N) \rightarrow \underline{\text{SO}}(\Omega_\bullet, N) \rightarrow 1,$$

which is exact in the *fppf* topology on  $\text{Spec}(\mathbb{Z})$ . We refer to the paper of Bass [3] for a precise definition of  $\underline{\text{Spin}}$  and  $\underline{\text{SO}}$  in this case – we do not simply use the determinant to define  $\underline{\text{SO}}$ , since that is not the correct notion in the characteristic 2 fibre. Following Proposition 4.8 of [27] we may realize  $\underline{\text{Spin}}$  as a subgroup scheme of  $\underline{\text{SO}}^3$ :

$$\underline{\text{Spin}}(\Omega_\bullet, N) = \{(\xi, v, \zeta) \in \underline{\text{SO}}(\Omega_\bullet, N)^3 : \text{Tr}(\xi\alpha^v\beta^\zeta\gamma) = \text{Tr}(\alpha\beta\gamma) \text{ for } \alpha, \beta, \gamma \in \Omega_\bullet\}.$$

We write  $\underline{\mathbf{G}}_\bullet$  for  $\underline{\text{Spin}}(\Omega_\bullet, N)$ , and view points of  $\underline{\mathbf{G}}_\bullet$  as triples as above. Following Gross [18],  $\underline{\mathbf{G}}_s$  is the unique model over  $\mathbb{Z}$  of the globally split, simply connected simple group of type  $D_4$ .  $\underline{\mathbf{G}}_c$  is the unique model over  $\mathbb{Z}$  of the simply connected simple group of type  $D_4$  which is split at every finite place  $p$ , and which is anisotropic at the real place.

The inclusion of  $\mu_2$  in the center of  $\underline{\text{SO}}(\Omega_\bullet, N)$  yields an inclusion of  $\mu_2^3$  in the center of  $\underline{\text{SO}}(\Omega_\bullet, N)^3$ . Define the group scheme:

$$\nu = \{(\xi, v, \zeta) \in \mu_2^3 : \xi v \zeta = 1\}.$$

If  $R$  is an integral domain of characteristic zero, then  $\nu(R)$  is the finite group  $(\mathbb{Z}/2\mathbb{Z})^2$ . Our description of  $\underline{\text{Spin}}(\Omega_\bullet, N)$  fixes a natural embedding of  $\nu$  in the center of  $\underline{\mathbf{G}}_\bullet$ .

This construction of  $\underline{\mathbf{G}}_\bullet$  can be used to construct the unique model over  $\mathbb{Z}$  of the simply connected group of type  $E_8$  which is split at every finite place  $p$ , and which is quaternionic at the real place. Let  $\mathfrak{g}_\bullet$  denote the Lie algebra over  $\mathbb{Z}$  of the group scheme  $\underline{\mathbf{G}}_\bullet$ , which for all rings  $R$  satisfies:

$$\mathfrak{g}_\bullet \otimes_{\mathbb{Z}} R \cong \ker(\underline{\mathbf{G}}_\bullet(R[\epsilon]/\epsilon^2) \rightarrow \underline{\mathbf{G}}_\bullet(R)).$$

Let  $\mathfrak{g}_d$  denote the Lie algebra over  $\mathbb{Z}$  of the group scheme  $(\underline{\mathbf{G}}_c \times_\nu \underline{\mathbf{G}}_s)$ , where the subscript  $\nu$  denotes the natural identification of the copies of  $\nu$  in the centers of  $\underline{\mathbf{G}}_c$  and  $\underline{\mathbf{G}}_s$ . Then  $\mathfrak{g}_d$  contains  $\mathfrak{g}_c \oplus \mathfrak{g}_s$  with index 4.

The three actions of  $\underline{\mathbf{G}}_\bullet$  on  $\Omega_\bullet$ , which exist by the construction of  $\underline{\mathbf{G}}_\bullet$  as a subgroup scheme of  $\underline{\text{SO}}(\Omega_\bullet)^3$ , yield three infinitesimal actions of the Lie algebra  $\mathfrak{g}_d$

on  $\Omega_c \otimes_{\mathbb{Z}} \Omega_s$ . The algebra structure on  $\Omega_c$  and  $\Omega_s$  can be further exploited to give a Lie algebra structure on the lattice:

$$\mathfrak{e}_{8,4} = \mathfrak{g}_d \oplus (\Omega_c \otimes_{\mathbb{Z}} \Omega_s) \oplus (\Omega_c \otimes_{\mathbb{Z}} \Omega_s) \oplus (\Omega_c \otimes_{\mathbb{Z}} \Omega_s).$$

This construction is completely described by Loke in [31].

The schematic closure of the subgroup of  $\underline{\mathbf{GL}}(\mathfrak{e}_{8,4})$  preserving a Killing form and the Lie bracket is an adjoint and simply connected group scheme  $\underline{\mathbf{E}}_{8,4}$  over  $\mathbb{Z}$ , which is split of type  $E_8$  over  $\mathbb{Q}_p$  at every finite place  $p$ , and which is the quaternionic real form  $E_{8,4}$  over  $\mathbb{R}$ . Essentially by construction, there is an inclusion:

$$\underline{\mathbf{G}}_c \times_{\nu} \underline{\mathbf{G}}_s \hookrightarrow \underline{\mathbf{E}}_{8,4}.$$

Now, we construct the group schemes  $\underline{\mathbf{E}}_{7,3}$  and  $\underline{\mathbf{E}}_{7,7}$  in a way that makes triality and the dual pair  $\underline{\mathbf{G}}_{\bullet} \times (\underline{\mathbf{SL}}_2^3)$  easy to see. Note that there is a natural embedding of  $\nu$  in the center of  $\underline{\mathbf{SL}}_2^3$ , so it makes sense to consider the group schemes  $\underline{\mathbf{G}}_{\bullet} \times_{\nu} \underline{\mathbf{SL}}_2^3$ .

Triples  $(m_1, m_2, m_3)$  in the Lie algebra  $\mathfrak{sl}_2$  act on triples  $(y_1, y_2, y_3) \in Y_s$  by writing:

$$(m_1, m_2, m_3) \cdot (y_1, y_2, y_3) = (m_3 y_1 - y_1 m_2, m_1 y_2 - y_2 m_3, m_2 y_3 - y_3 m_1).$$

This gives a natural action of  $\text{Lie}(\underline{\mathbf{G}}_{\bullet} \times_{\nu} \underline{\mathbf{SL}}_2^3)$  on  $(\Omega_{\bullet} \otimes Y_s)^3$ . Combined with the algebra structures on  $\Omega_{\bullet} \otimes Y_s$ , there are natural Lie algebra structures on the lattices:

$$\begin{aligned} \mathfrak{e}_{7,7} &= \text{Lie}(\underline{\mathbf{G}}_s \times_{\nu} \underline{\mathbf{SL}}_2^3) \oplus (\Omega_s \otimes_{\mathbb{Z}} Y_s) \oplus (\Omega_s \otimes_{\mathbb{Z}} Y_s) \oplus (\Omega_s \otimes_{\mathbb{Z}} Y_s), \\ \mathfrak{e}_{7,3} &= \text{Lie}(\underline{\mathbf{G}}_c \times_{\nu} \underline{\mathbf{SL}}_2^3) \oplus (\Omega_c \otimes_{\mathbb{Z}} Y_s) \oplus (\Omega_c \otimes_{\mathbb{Z}} Y_s) \oplus (\Omega_c \otimes_{\mathbb{Z}} Y_s). \end{aligned}$$

Again, we point to the work of Loke [31] for a detailed description of these Lie algebras over  $\mathbb{Z}$ .

The schematic closure of the subgroups of  $\underline{\mathbf{GL}}(\mathfrak{e}_{7,7})$  and  $\underline{\mathbf{GL}}(\mathfrak{e}_{7,3})$  preserving a Killing form and the Lie brackets are simply connected group schemes  $\underline{\mathbf{E}}_{7,7}$  and  $\underline{\mathbf{E}}_{7,3}$  over  $\mathbb{Z}$ , which are split of type  $E_7$  at every finite place, and have real rank 7 and 3 over  $\mathbb{R}$  respectively. Again, there are dual pair inclusions:

$$\begin{aligned} \underline{\mathbf{G}}_s \times_{\nu} \underline{\mathbf{SL}}_2^3 &\hookrightarrow \underline{\mathbf{E}}_{7,7}, \\ \underline{\mathbf{G}}_c \times_{\nu} \underline{\mathbf{SL}}_2^3 &\hookrightarrow \underline{\mathbf{E}}_{7,3}. \end{aligned}$$

## 2. MODULAR FORMS ON $\underline{\mathbf{G}}_s$

**2.1. The Heisenberg parabolic.** Choose a maximal torus  $\underline{\mathbf{T}}_s$  contained in a Borel subgroup  $\underline{\mathbf{B}}_s$  in  $\underline{\mathbf{G}}_s$ , all over  $\mathbb{Z}$ . Let  $\Delta_s$  denote the resulting set of simple roots for  $\underline{\mathbf{G}}_s$ . The root system of  $\underline{\mathbf{G}}_s$  is of type  $D_4$  and contains four simple roots

$$\Delta_s = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\},$$

with  $\alpha_0$  the “central” root, with single edges joining  $\alpha_0$  to  $\alpha_i$  for  $i = 1, 2, 3$  in the Dynkin diagram. The highest root  $\beta_0$  for  $\underline{\mathbf{G}}_s$  can be decomposed:

$$\beta_0 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_0.$$

Let  $\underline{\mathbf{P}}_s$  be the Heisenberg parabolic of  $\underline{\mathbf{G}}_s$ , associated to the subset of simple roots  $\{\alpha_1, \alpha_2, \alpha_3\}$ . Let  $\underline{\mathbf{L}}_s$  be a Levi component of  $\underline{\mathbf{P}}_s$  over  $\mathbb{Z}$  containing  $\underline{\mathbf{T}}_s$ . Let  $\underline{\mathbf{L}}'_s$  denote the derived subgroup of  $\underline{\mathbf{L}}_s$ . Then  $\underline{\mathbf{L}}'_s$  is isomorphic to  $\underline{\mathbf{SL}}_2^3$ . The unipotent radical  $\underline{\mathbf{H}}_s$  is a group of Heisenberg type, with center  $\underline{\mathbf{Z}}$  one-dimensional. We call the abelian unipotent 8-dimensional quotient  $\underline{\mathbf{H}}_s/\underline{\mathbf{Z}} = \underline{\mathbf{C}}$ . The representation of  $\underline{\mathbf{L}}'_s$  by conjugation on  $\underline{\mathbf{C}}$  is the tensor cube of the standard 2-dimensional representation

of  $\underline{\mathbf{SL}}_2$ . We view points of  $\underline{\mathbf{C}}$  as 2 by 2 by 2 cubes, and discuss these in a later section. For a ring  $R$ , an element of  $\underline{\mathbf{L}}_s(R)$  can be written as a triple of  $2 \times 2$  matrices with coefficients in  $R$ , such as  $l = (l_1, l_2, l_3)$ , with common non-zero determinant  $\det(l)$ .

**2.2. Quaternionic discrete series and their continuation.** We work over  $\mathbb{R}$ , and so we write  $G_s$  for the real Lie group  $\mathbf{G}_s(\mathbb{R}) \cong Spin_{4,4}(\mathbb{R})$ . Let  $\varrho$  denote half of the sum of the positive roots, and recall  $\beta_0$  is the highest root.

First, we review a few properties of the quaternionic discrete series representations of  $G_s$  after Gross and Wallach [21] and Wallach [37]. The maximal compact subgroup of  $G_s$  is  $K \cong SU(2) \times_{\mu_2} (SU(2) \times SU(2) \times SU(2))$ . The representations of  $G_s$  that we consider are classified by a lowest  $K$ -type; these in turn are classified by certain quadruples  $(k, \omega_1, \omega_2, \omega_3)$  of non-negative integers. Let  $\mathbb{C}^2$  denote the standard representation of  $SU(2)$  and for  $\omega = (\omega_1, \omega_2, \omega_3)$ , define the representation of  $SU(2) \times SU(2) \times SU(2)$ :

$$W_\omega = Sym^{\omega_1}(\mathbb{C}^2) \boxtimes Sym^{\omega_2}(\mathbb{C}^2) \boxtimes Sym^{\omega_3}(\mathbb{C}^2),$$

the external tensor product representation. We say that a pair  $(k, \omega)$  with  $\omega = (\omega_1, \omega_2, \omega_3)$  is *even* if  $k \geq 2$  and  $k + \omega_1 + \omega_2 + \omega_3$  is even. Descending representations from  $SU(2) \times SU(2) \times SU(2) \times SU(2)$  to  $K$  is described by:

**Proposition 2.1.** *There is a bijection between the set of even  $(k, \omega)$  and the set of irreducible representations of  $K$ . For every even pair  $(k, \omega)$ , we associate the representation of  $K$ :*

$$Sym^{k-2}(\mathbb{C}^2) \boxtimes W_\omega.$$

When  $\omega = 0$ , Gross and Wallach describe representations of  $G_s$  with lowest  $K$ -type  $Sym^{k-2}(\mathbb{C}^2) \boxtimes \mathbb{C}$  in [21]; more generally, we summarize some results mentioned in Loke (cf. Theorem 3.3.1 in [30]):

**Proposition 2.2.** *For  $9 \leq k \in \mathbb{Z}$ , and  $(k, \omega)$  even there is a “quaternionic” discrete series representation of  $G_s$  with infinitesimal character  $\varrho - \frac{k}{2}\beta_0$ , of Gelfand-Kirillov dimension 9, whose Casselman-Wallach globalization we denote  $\pi_{k,\omega}$ . The  $K$ -finite vectors in  $\pi_{k,\omega}$  decompose as a  $K$ -module via the representations:*

$$\bigoplus_{n \geq 0} Sym^{k-2+n}(\mathbb{C}^2) \boxtimes (Sym^n(W_{111}) \otimes W_\omega).$$

*Even for  $2 \leq k < 9$ , (with  $(k, \omega)$  still even) one may analytically construct representations like the  $\pi_{k,\omega}$ , which will not however be in the discrete series. Specifically, if  $2 \leq k$ , there are smooth representations  $\pi'_{k,\omega}$  of  $G_s$  on a complex Fréchet space, with finite length and infinitesimal character  $\varrho - \frac{k}{2}\beta_0$ , whose  $K$ -finite vectors still decompose according to the previous formula. This representation  $\pi'_{k,\omega}$  may be reducible, but it contains a unique irreducible sub-module  $\pi_{k,\omega}$  spanned by the  $G_s$ -translates of the lowest  $K$ -type.*

When  $\omega = 0$ , we write  $\pi_k$  instead of  $\pi_{k,\omega}$ . In particular, the representation  $\pi_2$  is unitarizable, and is the minimal representation studied by Kostant in [28] of Gelfand-Kirillov dimension 5.



**2.3. Modular forms.** This section is adapted from Section 7 of Gross, Gan, and Savin [15]. Fix  $(k, \omega)$  even, and let  $\pi_{k, \omega}$  denote the irreducible representation of  $\mathbf{G}_s(\mathbb{R})$  discussed in the last section (a discrete series representation if  $9 \leq k$ ).

Let  $\mathcal{A}_s = \mathcal{A}(\mathbf{G}_s)$  denote the space of automorphic forms on  $\mathbf{G}_s$ , as discussed in the background section.  $\mathcal{A}_s^0$  denotes the subspace of cuspidal automorphic forms.

**Definition 2.3.** Let  $\hat{K}$  denote an open compact subgroup of  $\mathbf{G}_s(\hat{\mathbb{Q}})$ . The space of weight  $(k, \omega)$  and level  $\hat{K}$  modular forms on  $\underline{\mathbf{G}}_s$  is defined to be:

$$\mathcal{M}_s(k, \omega, \hat{K}) = \text{Hom}_{\mathbf{G}_s(\mathbb{R}) \times \hat{K}}(\pi_{k, \omega} \boxtimes \mathbb{C}, \mathcal{A}_s).$$

When  $\omega = 0$ , we write  $\mathcal{M}_s(k, \hat{K})$ ; we think of  $\mathcal{M}_s(k, \hat{K})$  as a space of scalar-valued modular forms of weight  $k$ , whereas  $\mathcal{M}_s(k, \omega, \hat{K})$  is a space of vector-valued modular forms. The space of weight  $(k, \omega)$  and level 1 modular forms on  $\underline{\mathbf{G}}_s$  is defined to be:

$$\mathcal{M}_s(k, \omega, 1) = \text{Hom}_{\mathbf{G}_s(\mathbb{R}) \times \underline{\mathbf{G}}_s(\hat{\mathbb{Z}})}(\pi_{k, \omega} \boxtimes \mathbb{C}, \mathcal{A}_s).$$

The space of weight  $(k, \omega)$  cusp forms is defined likewise, replacing  $\mathcal{A}_s$  by  $\mathcal{A}_s^0$ , and is denoted  $\mathcal{M}_s^0(k, \omega, \hat{K})$ .

**2.4. 2 by 2 by 2 cubes.** In this section, we follow Bhargava [4], and study the scheme  $\underline{\mathbf{C}}$  over  $\mathbb{Z}$  which satisfies  $\underline{\mathbf{C}}(R) = R^2 \otimes R^2 \otimes R^2$  for any ring  $R$ . If  $c \in \underline{\mathbf{C}}(R)$ , then we think of  $c$  as a 2 by 2 by 2 cube  $c = (c_{i,j,k})$  of elements of  $R$ , with  $i, j, k \in \{0, 1\}$ . If  $c$  is a cube, then there are three faces  $F_1, F_2, F_3$  of  $c$  which contain the entry  $c_{0,0,0}$ . Let  $F'_i$  denote the faces opposite  $F_i$  for  $i = 1, 2, 3$ . We view the  $F_i$  and  $F'_i$  naturally as 2 by 2 matrices with coefficients in  $R$ . From these faces, we define 3 binary quadratic forms:

$$Q_i(x, y) = -\det(F_i x - F'_i y),$$

for  $i = 1, 2, 3$ . The discriminants of the three quadratic forms  $Q_i$  are equal to a single  $\Delta = \Delta(c)$ ; thus we call  $\Delta(c)$  the discriminant of  $c$ . We say that  $c$  is non-degenerate if  $\Delta(c) \neq 0$ .  $\Delta$  is a quartic polynomial map on  $\underline{\mathbf{C}}$  with integer coefficients.

The tensor cube of the standard representation yields a natural action of  $\underline{\mathbf{SL}}_2^3$  on  $\underline{\mathbf{C}}$ . The polynomial  $\Delta$  generates the polynomial invariants for this action. A cube  $c \in \underline{\mathbf{C}}(\mathbb{Z})$  is said to be projective if the three quadratic forms  $Q_1, Q_2, Q_3$  are primitive. In particular, if  $c$  is a projective cube, then the greatest common divisor of its entries  $c_{i,j,k}$  equals one. Following the appendix in [4], it is useful to know that projective cubes can be put into a particularly nice form via the action of  $\Gamma = \underline{\mathbf{SL}}_2(\mathbb{Z})^3$ :

**Proposition 2.4.** *Every projective cube  $c \in \underline{\mathbf{C}}(\mathbb{Z})$  is  $\Gamma$ -conjugate to some cube  $\tilde{c}$  which satisfies  $\tilde{c}_{0,0,0} = 1$ , and  $\tilde{c}_{1,0,0} = \tilde{c}_{0,1,0} = \tilde{c}_{0,0,1} = 0$ .*

A cube  $c$  which already satisfies the conditions in the above proposition is said to be in *normal form*. The discriminant of a cube  $c$  in normal form is given by:

$$\Delta(c) = c_{1,1,1}^2 + 4c_{1,1,0}c_{1,0,1}c_{0,1,1}.$$

For any integer  $D \equiv 0, 1 \pmod{4}$ , let  $R(D)$  denote the unique quadratic ring over  $\mathbb{Z}$  having discriminant  $D$  given by:

$$\begin{cases} \mathbb{Z}[x]/(x^2) & \text{if } D = 0, \\ \mathbb{Z} + \sqrt{D}(\mathbb{Z} \oplus \mathbb{Z}) & \text{if } D \geq 1, \sqrt{D} \in \mathbb{Z}, \\ \mathbb{Z}[(D + \sqrt{D})/2] & \text{otherwise.} \end{cases}$$

An *invertible oriented ideal* in  $R(D)$  is a pair  $(I, \epsilon)$  where  $I$  is an invertible fractional ideal of  $R(D)$  in  $R(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ , and  $\epsilon = \pm 1$ . These form an abelian group by component-wise multiplication. An invertible *principal* oriented ideal is one of the form  $((k), \text{sgn}(N(k)))$ , where  $(k)$  is the principal ideal generated by  $k \in R(D)$  which is invertible in  $R(D) \otimes_{\mathbb{Z}} \mathbb{Q}$ , and  $N(k)$  is the norm of  $k$  in  $\mathbb{Z}$ . The *narrow ideal class group* of  $R(D)$  is defined to be the quotient group of invertible oriented ideals modulo invertible principal oriented ideals. When  $D > 0$  is square-free, this agrees with the narrow ideal class group of real quadratic fields. When  $D < 0$  is squarefree, the norm form on  $R(D)$  is positive-definite, and the narrow ideal class group contains the usual ideal class group with index 2.

A result essentially known to Gauss is the following:

**Proposition 2.5.** *The set of  $SL_2(\mathbb{Z})$  orbits on the set of non-degenerate primitive binary quadratic forms is in bijection with the set of pairs  $(D, I)$ , where  $D \neq 0$  is a discriminant, i.e., a positive integer congruent to 0 or 1 mod 4, and where  $I$  is in the narrow ideal class group of  $R(D)$ .*

A beautiful generalization of the above result is a consequence of Theorem 11 in Bhargava [4]:

**Proposition 2.6.** *The set of  $SL_2(\mathbb{Z})^3$  orbits on the set of primitive non-degenerate integer cubes of discriminant  $D \neq 0$  is in bijection with the set of triples  $(I_1, I_2, I_3)$ , where  $(I_1, I_2, I_3)$  is a triple of narrow ideal classes in  $R(D)$  whose product  $I_1 \cdot I_2 \cdot I_3$  is the principal class.*

We have seen part of this proposition, since each such integer cube yields three primitive quadratic forms of the same discriminant  $D$ , and hence three narrow ideal classes in  $R(D)$  by the previous result.

**2.5. Heisenberg Whittaker models.** Heisenberg Whittaker models of quaternionic discrete series provide the foundation for the Fourier expansion of modular forms in  $\mathcal{M}_s$ . We review the theory of these Whittaker models here. As we work only over  $\mathbb{R}$  for the moment, we write  $H_s, C, Z, L_s$  for  $\mathbf{H}_s(\mathbb{R}), \mathbf{C}(\mathbb{R}), \mathbf{Z}(\mathbb{R}), \mathbf{L}_s(\mathbb{R})$ . Recall that  $H_s$  is a Heisenberg group of dimension 9, with abelian quotient  $C = H_s/Z$ . The non-degenerate  $Z$ -valued symplectic form on  $H_s$  allows us to identify  $C$  with the group of characters  $\text{Hom}(H_s, S^1)$  in such a way that  $\underline{\mathbf{C}}(\mathbb{Z})$  is identified with the set of characters which are trivial on  $\underline{\mathbf{H}}_s(\mathbb{Z})$ . Hence, for any  $c \in C$ , we let  $\mathbb{C}_c$  denote the set of complex numbers, viewed as a representation of  $H_s$  via the character  $\chi_c$  associated to  $c$ . Let  $\Delta$  denote the discriminant function on  $C$ , which is a quartic polynomial. For any such  $c$ , and for any even  $(k, \omega)$ , let  $Wh_{k, \omega}(c)$  denote the space of Whittaker models:

$$Wh_{k, \omega}(c) = \text{Hom}_{H_s}(\pi_{k, \omega}, \mathbb{C}_c).$$

Recall that the Levi component  $L_s$  can be identified with the group of triples  $l = (l_1, l_2, l_3)$  with  $l_i \in GL_2(\mathbb{R})$ , such that  $\det(l_1) = \det(l_2) = \det(l_3)$ . Thus we write

$\det(l)$  for this common determinant. The Levi component  $L_s$  acts by conjugation on  $H_s$ , preserving the symplectic form up to scalar, and thus acting on  $C$ . As a result,  $L_s$  acts on objects related to  $C$ ; if  $l \in L_s$  and  $c \in C$  then:

- We define  $l[c] \in C(\mathbb{R})$  by  $l[c] = lcl^{-1}$ .
- $\chi_{l[c]}(h) = \chi_c(l^{-1}hl)$  gives an action of  $l$  on characters of  $H_s$ .
- If  $w \in Wh_{k,\omega}(c)$ , and  $v$  is in the space of  $\pi_{k,\omega}$ , we write  $l[w](v) = w(l^{-1}(v))$ . Then  $l[w] \in Wh_{k,\omega}(l[c])$ .

In particular, there is a natural action of the stabilizer of  $c$  in  $L_s$ ,  $L_{s,c}$ , on the space  $Wh_{k,\omega}(c)$ .

Let  $W_{k,\omega}$  denote the representation of  $L_s$  of highest weight  $k\alpha_0 + \omega_1\alpha_1 + \omega_2\alpha_2 + \omega_3\alpha_3$ , and  $\tilde{W}_{k,\omega}$  its contragredient. From Theorem 16 of Wallach [37] (and coherent continuation for  $k < 9$ ), we have:

**Proposition 2.7.** *Suppose  $c \in C$ . If  $\Delta(c) < 0$ , then  $Wh_{k,\omega}(c)$  is isomorphic to  $\tilde{W}_{k,\omega}$  as a representation of the stabilizer  $L_{s,c}$  of  $c$  in  $L_s$ . If  $\Delta(c) > 0$ , then  $Wh_{k,\omega}(c) = 0$ . In particular, if  $\omega = 0$ , then  $Wh_k(c)$  is 1-dimensional if  $\Delta(c) < 0$  and the action is  $L_{s,c}$  is given by  $\det^{-3k}$ .*

*Remark 2.8.* It is difficult to ensure that the notion of “admissible characters” used by Wallach in [37] agrees with the sign of the discriminant used above; such sign errors are very easy to make. Following advice of W.-T. Gan, we have worked backwards to deduce the correct sign from the existence of a Siegel-Weil type embedding problem. Namely, we expect the Fourier coefficients of a certain modular form to count embeddings of quadratic rings into the (non-split) octonions with some extra structure. Such embeddings are only possible if the quadratic rings are imaginary, and thus if the sign of  $\Delta$  in the above theorem is negative.

**2.6. Fourier Expansion.** We consider the Fourier expansion of level 1 modular forms on  $\underline{\mathbf{G}}_s$ . Since  $\underline{\mathbf{G}}_s$  is a simply-connected simple algebraic group, strong approximation holds, and we may identify:

$$\mathbf{G}_s(\mathbb{Q}) \backslash \mathbf{G}_s(\mathbb{A}) / \mathbf{G}_s(\hat{\mathbb{Z}}) \cong \underline{\mathbf{G}}_s(\mathbb{Z}) \backslash \mathbf{G}_s(\mathbb{R}).$$

For any vector  $v \in \pi_{k,\omega}$ , and any  $f \in \mathcal{M}_s(k, \omega, 1)$ , we write  $f_v = f(v)$ , and view  $f_v$  as a function on the single coset space  $\underline{\mathbf{G}}_s(\mathbb{Z}) \backslash \mathbf{G}_s(\mathbb{R})$ . Let  $\chi_c$  denote any character of the unipotent radical  $\mathbf{H}_s(\mathbb{R})$  of the Heisenberg parabolic, which is trivial on  $\underline{\mathbf{H}}_s(\mathbb{Z})$ , associated to an element  $c$  of  $\underline{\mathbf{C}}(\mathbb{Z})$ . For any continuous function  $f$  on  $\underline{\mathbf{H}}_s(\mathbb{Z}) \backslash \mathbf{H}_s(\mathbb{R})$ , define the (abelian) Fourier coefficient of  $f$  by:

$$\mathcal{F}_c(f) = \int_{\underline{\mathbf{H}}_s(\mathbb{Z}) \backslash \mathbf{H}_s(\mathbb{R})} f(h) \overline{\chi_c(h)} dh.$$

We may define the linear form  $w_c \in Wh_{k,\omega}(c)$  by:

$$w_c(v) = \mathcal{F}_c(f_v).$$

Proposition 8.2 of [15] carries over to our case to verify that  $w_c$  lies in  $Wh_{k,\omega}(c)$ . In particular,  $w_c = 0$  if  $\Delta(c) > 0$ . Moreover if  $c' = \gamma[c]$ , with  $\gamma \in \underline{\mathbf{L}}_s(\mathbb{Z})$ , then we have:

$$w_{c'}(v) = \gamma[w_c](v).$$

When  $f$  is a scalar-valued modular form of level 1,  $f \in \mathcal{M}_s(k, 1)$ , the uniqueness of Whittaker models in Proposition 2.7 can be exploited to get scalar Fourier coefficients of  $f$  indexed by cubes. The set of  $c \in C(\mathbb{R})$  with  $\Delta(c) < 0$  forms a single

$\mathbf{L}_s(\mathbb{R})$  orbit, and we fix an arbitrary  $c_0$  in this orbit. Since  $f$  is scalar-valued, we may also choose an element  $w_0 \in Wh_k(c_0)$  which spans  $Wh_k(c_0)$ . Hence for any other  $c$  of negative discriminant,  $w_c$  is a scalar multiple of  $l[w_0]$  for some  $l \in \mathbf{L}_s(\mathbb{R})$ , well-defined up to the stabilizer of  $c$  in  $\mathbf{L}_s(\mathbb{R})$ . As a result, there are well-defined constants  $a_c$  for all  $c$  of negative discriminant such that:

$$w_c = a_c \cdot \det(l)^{-3k} l[w_0].$$

The factor of  $\det(l)^{-3k}$  is crucial to make the constant  $a_c$  well-defined, even though  $l$  is well-defined only up to  $L_{s,c}$ , using Proposition 2.7.

When  $f$  is a vector-valued modular form of weight  $(k, \omega)$  and of level 1, this may be appropriately generalized: the coefficients  $a_c$  will no longer be scalars, but will have matrix values in  $\text{End}(W_{k,\omega})$ . When  $f$  is scalar-valued and of level 1, we have:

**Theorem 2.9.** *Suppose  $f \in \mathcal{M}_s(k, 1)$ . Then the Fourier expansion of  $f$  along the Heisenberg parabolic yields constants  $a_c$  for every 2 by 2 by 2 cube  $c$ . These constants are well-defined, up to a uniform scaling. The constants  $a_c$  vanish if  $\Delta(c) > 0$ . For  $\Delta(c) < 0$ , the constant  $a_c$  depends only on the  $\Gamma$ -orbit of  $c$ . Hence, we can associate to each quadruple  $(D, I_1, I_2, I_3)$  (a negative discriminant  $D = \Delta(c)$  and three oriented ideal classes whose product is the principal class) a constant  $a_{(D, I_1, I_2, I_3)}$ .*

Let  $NCl(D)$  denote the narrow class group of the quadratic ring  $R(D)$ . The structure of  $NCl(D)$  is completely described by the graph of its group law, i.e., the set of triples  $I_1, I_2, I_3$  in  $NCl(D)$  whose product is 1. Thus modular forms  $f \in \mathcal{M}_s(k, 1)$  may provide information on the fine structure of ideal class groups of imaginary quadratic fields.

### 3. MODULAR FORMS ON $\mathbf{G}_c$

We now consider modular forms on the group  $\mathbf{G}_c$ ; recall that  $\mathbf{G}_c(\mathbb{Q}_p)$  is the split simply-connected simple group of type  $D_4$  for all  $p$ , and  $\mathbf{G}_c(\mathbb{R})$  is isomorphic to the compact simply-connected simple group  $Spin_8(\mathbb{R})$ . Thus we may study modular forms on  $\underline{\mathbf{G}}_c$  in the framework of Gross's "algebraic modular forms" [19]. With the integral model  $\underline{\mathbf{G}}_c$  constructed in the first section, the group of integer points  $\underline{\mathbf{G}}_c(\mathbb{Z})$  has cardinality  $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ , and is isomorphic to the finite group  $\Gamma_c := 2^2 \cdot O_8^+(2)$  [18]. The group  $\underline{\mathbf{G}}_c(\mathbb{Z})$  is a central extension of its reduction  $\underline{\mathbf{G}}_c(\mathbb{F}_2)$  by the subgroup  $\nu(\mathbb{Z})$  of order 4. More generally, every arithmetic subgroup of  $\mathbf{G}_c(\mathbb{Q})$  is finite.

**3.1. Algebraic modular forms.** Since  $\mathbf{G}_c$  is an inner form of the split group  $\mathbf{G}_s$  over  $\mathbb{Q}$ , the irreducible algebraic representations of  $\mathbf{G}_c$  over  $\mathbb{Q}$  are parameterized by dominant weights (over an algebraic closure); these are parameterized by non-negative pairs  $(k, \omega)$ , with  $\omega = (\omega_1, \omega_2, \omega_3)$ . Let  $V_{k,\omega}$  denote the associated algebraic representation of  $\mathbf{G}_c$  on a rational vector space.

Let  $\mathcal{A}_c = \mathcal{A}(\mathbf{G}_c)$  denote the space of automorphic forms on  $\mathbf{G}_c$ ; for the definition, we refer the reader to the background section. The condition of moderate growth is unnecessary here, since  $\mathbf{G}_c(\mathbb{R})$  is compact. Fix an open compact subgroup  $\hat{K}$  of  $\mathbf{G}_c(\hat{\mathbb{Q}})$ ; note that  $\mathbf{G}_c(\hat{\mathbb{Q}}) \cong \mathbf{G}_s(\hat{\mathbb{Q}})$  so the choices of level structure for  $\mathbf{G}_c$  and  $\mathbf{G}_s$  are equivalent. The space of modular forms of weight  $(k, \omega)$  and level  $\hat{K}$  is then:

$$\mathcal{M}_c(k, \omega, \hat{K}) = \text{Hom}_{\mathbf{G}_c(\mathbb{R}) \times \hat{K}}(V_{k,\omega} \boxtimes \mathbb{C}, \mathcal{A}_c).$$

When  $\omega = 0$ , we write  $\mathcal{M}_c(k, \hat{K})$ , and when  $\hat{K} = \underline{\mathbf{G}}_c(\hat{\mathbb{Z}})$ , we write  $\mathcal{M}_c(k, \omega, 1)$  and view these as modular forms of level 1. The complex vector space  $\mathcal{M}_c(k, \omega, \hat{K})$  has a natural rational structure, through Gross's theory of "algebraic modular forms" in [19]. We describe this algebraic structure here:

Define the  $\mathbb{Q}$  vector space  $\mathcal{M}_c^{alg}(k, \omega, \hat{K})$  to be the set of functions  $f$  from  $\mathbf{G}_c(\mathbb{A})$  to  $V_{k, \omega}$  which are right-invariant by  $\hat{K}$ , left-*equivariant* by  $\mathbf{G}_c(\mathbb{Q})$  (with its action on  $V_{k, \omega}$ ), and right-invariant by  $\mathbf{G}_c(\mathbb{R})$  too. Since  $\mathbf{G}_c$  is isomorphic to  $\mathbf{G}_s$  over  $\mathbb{Q}_p$  for all  $p$ , the space of modular forms  $\mathcal{M}_c(k, \omega, \hat{K})$  possesses a natural action of the same Hecke algebra  $\mathcal{H}(\hat{K})$  that acts on  $\mathcal{M}_s(k, \omega, \hat{K})$ .

For any  $f \in \mathcal{M}_c^{alg}(k, \omega, \hat{K})$ , we define an element  $F$  of  $\mathcal{M}_c(k, \omega, \hat{K})$  by writing:

$$F_v(g) = \langle g_\infty^{-1} f(g), v \rangle,$$

for all  $v \in V_{k, \omega} \otimes \mathbb{C}$  and  $g \in \mathbf{G}_c(\mathbb{A})$  with archimedean component  $g_\infty$ . From Proposition 8.5 of [19], we have:

**Proposition 3.1.** *The map  $f \mapsto F$  extends to a  $\mathcal{H}(\hat{K})$ -equivariant isomorphism:*

$$\mathcal{M}_c^{alg}(k, \omega, \hat{K}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathcal{M}_c(k, \omega, \hat{K}).$$

Thus modular forms on  $\underline{\mathbf{G}}_c$  can be studied purely algebraically. If the  $p$  component of  $\hat{K}$  is  $\underline{\mathbf{G}}_c(\mathbb{Z}_p)$ , then the local spherical Hecke algebra  $\mathcal{H}(K_p)$  acts on the space of modular forms; in this setting, it follows from Proposition 8.9 of [19] that every eigenvalue  $\lambda$  of a Hecke operator in  $\mathcal{H}(K_p)$  on  $\mathcal{M}_c(k, \omega, \hat{K})$  is algebraic; in fact, it lies in the ring of integers of a CM-field, localized away from  $p$ .

We study modular forms of level 1 in more detail. The double-coset space  $\mathbf{G}_c(\mathbb{Q}) \backslash \mathbf{G}_c(\mathbb{A}) / \mathbf{G}_c(\mathbb{R}) \underline{\mathbf{G}}_c(\hat{\mathbb{Z}})$  has only one element; this essentially follows from the uniqueness of the  $E_8$  lattice as an even unimodular positive-definite lattice of rank 8. From Proposition 4.5 of [19], we have:

**Proposition 3.2.** *The space of modular forms of weight  $(k, \omega)$  and level 1 is isomorphic to  $V_{k, \omega}^{\Gamma_c}$ .*

As a trivial first case, there is a one-dimensional  $\mathbb{Q}$ -vector space of modular forms of level 1 corresponding to the trivial representation  $V = \mathbb{Q}$  of  $\mathbf{G}_c(\mathbb{Q})$ . But  $\Gamma_c$  acts irreducibly on the three fundamental 8-dimensional representations of  $\mathbf{G}_c(\mathbb{Q})$ , so there are no modular forms of level 1 of weights  $(0, (1, 0, 0))$ ,  $(0, (0, 1, 0))$ , or  $(0, (0, 0, 1))$ . Of course, there will be modular forms of higher level corresponding to these representations, since the trivial group is an arithmetic subgroup of  $\mathbf{G}_c(\mathbb{Q})$ .

More generally, if  $\hat{K}$  is any level, we fix representatives  $g_\delta \in \mathbf{G}_c(\mathbb{A})$  for the finite collection of double-cosets  $\mathbf{G}_c(\mathbb{Q}) \backslash \mathbf{G}_c(\mathbb{A}) / \mathbf{G}_c(\mathbb{R}) \hat{K}$ . For each  $\delta$ , we have a finite arithmetic subgroup of  $\mathbf{G}_c(\mathbb{Q})$ :

$$\Gamma_\delta = \mathbf{G}_c(\mathbb{Q}) \cap g_\delta (\mathbf{G}_c(\mathbb{R}) \times \hat{K}) g_\delta^{-1}.$$

By Proposition 4.5 of [19], the choice of  $g_\delta$  yields an isomorphism, sending  $f$  to  $\bigoplus f(g_\delta)$ :

$$\mathcal{M}_c^{alg}(k, \omega, \hat{K}) \cong \bigoplus_{\delta} V_{k, \omega}^{\Gamma_\delta}.$$

**3.2. Some geometry for  $\mathbf{G}_c$ .** We have seen that there are three inequivalent 8-dimensional representations  $V_{0,(1,0,0)}, V_{0,(0,1,0)}, V_{0,(0,0,1)}$  of  $\mathbf{G}_c$ , all defined over  $\mathbb{Q}$ . Let  $\mathbb{O}_{\mathbb{Q}} = \Omega_c \otimes_{\mathbb{Z}} \mathbb{Q}$  denote the rational octonion division algebra. Then we view  $\mathbb{O}_{\mathbb{Q}}^3$  as the direct sum of the three 8-dimensional representations.

We examine the stabilizers in  $\mathbf{G}_c$  of various special triples in  $\mathbb{O}_{\mathbb{Q}}^3$ . Consider first the case where all but one of  $\alpha, \beta, \gamma$  equals 0. Let  $\mathbf{G}_I(\alpha), \mathbf{G}_{II}(\beta), \mathbf{G}_{III}(\gamma)$  denote the algebraic subgroups of  $\mathbf{G}_c$  stabilizing the triples  $(\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)$  respectively. All three of these groups become isomorphic to  $Spin_7$  over  $\mathbb{R}$ . Thus we call such groups  $Spin_7$  subgroups of  $\mathbf{G}_c$  of class  $I, II, III$ . By [36], there exist precisely three conjugacy classes of subgroups of the real Lie group  $Spin_8(\mathbb{R})$  isomorphic to  $Spin_7(\mathbb{R})$  – the real points of  $\mathbf{G}_I(\alpha), \mathbf{G}_{II}(\beta), \mathbf{G}_{III}(\gamma)$  represent these three conjugacy classes.

Now, consider the case when only one of  $\alpha, \beta, \gamma$  vanishes. Then the stabilizer of such a vector, e.g.  $(\alpha, \beta, 0)$ , in  $\mathbf{G}_c$  is a rational algebraic subgroup, which we call  $\mathbf{G}_{I,II}(\alpha, \beta)$ , which is an intersection of two  $Spin_7$  subgroups of different class. By Theorem 5 of the third section in [36], the intersection of two “unlike”  $Spin_7(\mathbb{R})$  subgroups in  $Spin_8(\mathbb{R})$  is a subgroup isomorphic to the compact Lie group  $G_2$ . Thus we call  $\mathbf{G}_{I,II}(\alpha, \beta)$  a  $G_2$  subgroup of  $\mathbf{G}_c$ . We have the following rational version of Theorem 5 of [36]:

**Proposition 3.3.** *Let  $\mathbf{G}_I(\alpha), \mathbf{G}_{II}(\beta)$  denote two  $Spin_7$  subgroups of  $\mathbf{G}_c$ . Then there exists a  $Spin_7$  subgroup of class  $III$ ,  $\mathbf{G}_{III}(\gamma)$  such that:*

$$\mathbf{G}_{I,II}(\alpha, \beta) = \mathbf{G}_I(\alpha) \cap \mathbf{G}_{II}(\beta) = \mathbf{G}_I(\alpha) \cap \mathbf{G}_{II}(\beta) \cap \mathbf{G}_{III}(\gamma).$$

*Proof.* Let  $\gamma = \overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$  (or any real scalar multiple thereof). If  $g = (\xi, v, \zeta) \in \mathbf{G}_{I,II}(\alpha, \beta)$ , then  $\xi$  stabilizes  $\alpha$  and  $v$  stabilizes  $\beta$ . To prove the proposition, it suffices to show that  $\zeta$  stabilizes  $\gamma$ ; in this case  $\mathbf{G}_{III}(\gamma) \supset \mathbf{G}_{I,II}(\alpha, \beta)$  and we are done.

To see that  $\zeta$  stabilizes  $\gamma$ , note that  $Tr(\alpha\beta\gamma) = Tr(\alpha\beta^{\zeta}\gamma)$  from the definition of  $\mathbf{G}_c$ . We see that  $\alpha\beta\gamma = N(\alpha\beta) \in \mathbb{Q}$ , and moreover  $\alpha\beta^{\zeta}\gamma$  is totally imaginary since it has zero trace. Hence the two vectors  $\alpha\beta\gamma$  and  $\alpha\beta^{\zeta}\gamma$  are perpendicular in  $\mathbb{O}_{\mathbb{Q}}$ . Note that  $N(\alpha\beta\gamma) = N(\alpha\beta^{\zeta}\gamma)$  and by the Pythagorean Theorem we see that  $N(\alpha\beta^{\zeta}\gamma - \gamma) = 0$ . Thus  $\zeta\gamma - \gamma = 0$ . Hence  $\gamma$  is stabilized by  $\zeta$  and we are done.  $\square$

We have seen that there are essentially three types of  $Spin_7$  subgroups of  $\mathbf{G}_c$ , which we called class  $I, II, III$ . However there is only one type of  $G_2$  subgroup of  $\mathbf{G}_c$  by the above proposition – they arise as intersections of two  $Spin_7$  subgroups of different classes, or equivalently as intersections of three  $Spin_7$  subgroups which are incident as in the proposition. Thus we define:

**Definition 3.4.** Suppose that  $(\alpha, \beta, \gamma) \in \mathbb{O}_{\mathbb{Q}}^3$  satisfies  $(\alpha\beta)\gamma \in \mathbb{Q}$ . Let  $\mathbf{G}_2(\alpha, \beta, \gamma)$  denote the stabilizer of the triple  $(\alpha, \beta, \gamma)$  in  $\mathbf{G}_c$ .

Finally, we consider the case of “generic” triples  $(\alpha, \beta, \gamma)$ , in the sense that  $(\alpha\beta)\gamma \notin \mathbb{Q}$ . The stabilizer in  $\mathbf{G}_c$  of such a triple is a subgroup, which by [36] is isomorphic to  $SU(3)$  over  $\mathbb{R}$ . Thus we define:

**Definition 3.5.** If  $\alpha, \beta, \gamma$  are non-zero octonions satisfying  $(\alpha\beta)\gamma \notin \mathbb{Q}$ , then let  $\mathbf{SU}_3(\alpha, \beta, \gamma)$  denote the rational algebraic subgroup of  $\mathbf{G}_c$  stabilizing the triple  $(\alpha, \beta, \gamma)$ .

**3.3. Periods.** Suppose that  $f \in \mathcal{M}_c^{alg}(k, \omega, \hat{K})$  is an algebraic modular form. Let  $\mathbf{G}'$  be any rational reductive subgroup of  $\mathbf{G}_c$ . The (vector-valued) period of  $f$  along  $\mathbf{G}'$  is defined to be:

$$\mathcal{P}_f^{\mathbf{G}'}(g) = \oint_{\mathbf{G}'} (g'_\infty)^{-1} f(g'g) dg.$$

We make the following definition:

**Definition 3.6.** The modular form  $f$  is  $\mathbf{G}'$ -distinguished if its  $\mathbf{G}'$ -period  $\mathcal{P}_f^{\mathbf{G}'}$  is non-zero.

As a trivial first case,  $f$  is not  $\mathbf{G}_c$ -distinguished if and only if  $f$  is orthogonal to all constant functions. Identifying modular forms with vectors in  $V_{k,\omega}^\Gamma$  for various finite groups  $\Gamma$  yields the following result:

**Proposition 3.7.** *Suppose  $f$  is a modular form of weight  $(k, \omega)$  and level  $\hat{K}$ , corresponding to  $v \in V_{k,\omega}^{\Gamma_\delta} = f(g_\delta)$ . If  $V_{k,\omega}$  has no non-zero vectors invariant under  $\mathbf{G}'(\mathbb{R})$ , then the period of  $f$  must vanish.*

*Proof.* The period map for a subgroup  $\mathbf{G}'$  of  $\mathbf{G}_c$  yields a  $\mathbf{G}'(\mathbb{R})$ -invariant functional on the infinity-type  $V_{k,\omega}$  of a modular form. Hence it is easy to see that if there are no  $\mathbf{G}'(\mathbb{R})$ -invariant vectors in  $V_{k,\omega}$ , this functional must vanish.  $\square$

Another criterion for  $f$  to have vanishing  $\mathbf{G}_2$  periods is given by Corollary 3.1 of [16], which we recall here:

**Proposition 3.8.** *Every irreducible admissible generic representation of  $\mathbf{G}_c(\mathbb{Q}_p)$  is not  $\mathbf{G}_2$ -distinguished, i.e., has no  $\mathbf{G}_2(\mathbb{Q}_p)$ -invariant functionals.*

Hence if  $f$  is a modular form on  $\mathbf{G}_c$ , which is generic at some finite place  $p$ , then  $f$  is not  $\mathbf{G}_2$ -distinguished.

Non-vanishing of periods is more difficult to prove than vanishing. In the level one case, some non-vanishing results can be proven without too much difficulty. Consider the triples  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$  of octonions first. The stabilizers of these triples are  $Spin_7$  subgroup schemes  $\underline{\mathbf{G}}_I(1, 0, 0)$ ,  $\underline{\mathbf{G}}_{II}(0, 1, 0)$ ,  $\underline{\mathbf{G}}_{III}(0, 0, 1)$  in  $\underline{\mathbf{G}}_c$ . The stabilizer of the triple  $(1, 1, 1)$  in  $\underline{\mathbf{G}}_c$  is a group scheme  $\underline{\mathbf{G}}_2(1, 1, 1)$ . All of these subgroup schemes have good reduction everywhere – they are good integral models of the simply connected simple groups of type  $B_3$  and  $G_2$ .

Also, fix a square root of  $-1$ ,  $j$ , in the integral octonions  $\Omega_c$ . The stabilizer of the triple  $(1, 1, j)$  is an  $SU_3$  subgroup scheme  $\underline{\mathbf{SU}}_3(1, 1, j)$  in  $\underline{\mathbf{G}}_c$ . We begin with:

**Lemma 3.9.** *The unitary group  $\underline{\mathbf{SU}}_3(1, 1, j)$  has class number 1, i.e., it satisfies:*

$$\#\{\mathbf{SU}_3(1, 1, j)(\mathbb{Q}) \backslash \mathbf{SU}_3(1, 1, j)(\hat{\mathbb{Q}}) / \underline{\mathbf{SU}}_3(1, 1, j)(\hat{\mathbb{Z}})\} = 1.$$

*Proof.* We describe the unitary group scheme  $\underline{\mathbf{SU}}_3(1, 1, j)$  more explicitly. Let  $\mathbb{F}$  denote the oriented “Fano plane”: the finite projective plane with 7 points, and 7 lines. Fix a basis  $\{1, e_0, e_1, \dots, e_6\}$  of  $\mathbb{O}_\mathbb{Q}$ , in which  $e_i^2 = -1$  for all  $0 \leq i \leq 6$ , and where we identify  $e_0, \dots, e_6$  with points  $p_0, \dots, p_6$  of  $\mathbb{F}$ . If  $i, j, k \in \mathbb{O}_\mathbb{Q}$  are square roots of  $-1$ , we call  $(i, j, k)$  a “quaternion triple” if they satisfy the familiar relations  $ij = -ji = k$ . The basis of  $\mathbb{O}_\mathbb{Q}$  may be chosen so that  $(e_i, e_j, e_k)$  is a quaternion triple in  $\mathbb{O}_\mathbb{Q}$  if  $p_i, p_j, p_k$  are oriented collinear points on  $\mathbb{F}$ . This fully describes the multiplication table of  $\mathbb{O}_\mathbb{Q}$  with respect to the chosen basis.

The basis elements  $\{1, e_0, \dots, e_6\}$  are contained in Coxeter’s order  $\Omega_c$ , though they do not span it. Identifying  $j$  with  $e_0$  (all square roots of  $-1$  are conjugate

under  $\underline{\mathbf{G}}_c$ ), let  $D$  denote the sublattice of  $\Omega_c$  orthogonal to  $\mathbb{Z}[e_0]$ . As a lattice,  $D$  is isomorphic to the  $D_6$  root lattice, and we describe  $D$  as:

$$D = \mathbb{Z}\text{-span} \left\{ \frac{1}{2}(e_i \pm e_j) \right\}, \text{ for } 1 \leq i, j \leq 6.$$

$D$  is closed under (left) multiplication by  $\mathbb{Z}[e_0]$ , and as a  $\mathbb{Z}[e_0]$ -module it is free of rank three with basis:

$$b_1 = \frac{e_1 + e_3}{2}, b_2 = \frac{e_2 + e_4}{2}, b_3 = \frac{e_5 - e_6}{2}.$$

On  $D$  there is a natural  $\mathbb{Z}[e_0]$ -valued Hermitian form given by:

$$h(d_1, d_2) = -d_1 d_2 + e_0(d_1 d_2) e_0.$$

An easy calculation shows that  $h(b_i, b_j) = \delta_{ij}$ , so that  $D$  is the simplest possible Hermitian lattice of rank 3 over  $\mathbb{Z}[e_0]$ . The group scheme  $\underline{\mathbf{SU}}_3(1, 1, j)$  is precisely the unitary group scheme of  $D$ . The lemma now follows directly from a result of K. Iyanaga [23] (who shows precisely that this unitary group scheme has class number 1).  $\square$

In the level 1 case, we can now prove that some periods do not vanish:

**Theorem 3.10.** *Suppose that  $f$  is an algebraic modular form of level 1 and weight  $(k, \omega)$ , corresponding to  $v \in V_{k, \omega}^{\Gamma_c}$ . Then if  $v$  is invariant under  $\mathbf{G}'(\mathbb{R})$  where  $\mathbf{G}'$  is one of  $\mathbf{G}_I(1, 0, 0)$ ,  $\mathbf{G}_{II}(0, 1, 0)$ ,  $\mathbf{G}_{III}(0, 0, 1)$ ,  $\mathbf{G}_2(1, 1, 1)$ , or  $\underline{\mathbf{SU}}_3(1, 1, j)$ , then the period  $\mathcal{P}_f^{\mathbf{G}'}$  is not zero.*

*Proof.* By the analysis in Proposition 5.5 of [20], to prove our theorem it suffices to show that:

$$\mathbf{G}'(\mathbb{A}) = \mathbf{G}'(\mathbb{Q})\mathbf{G}'(\mathbb{R})\underline{\mathbf{G}}'(\hat{\mathbb{Z}}).$$

For  $\mathbf{G}'$  of type  $Spin_7$  or  $G_2$ , this follows from the uniqueness of integral models discussed in [18]. For  $\mathbf{G}'$  of type  $SU_3$ , we apply the previous lemma, which shows that the particular unitary group  $\underline{\mathbf{SU}}_3(1, 1, j)$  has class number 1.  $\square$

**3.4. Some branching rules.** In order to determine whether periods of modular forms on  $\mathbf{G}_c$  vanish or not, it suffices by Proposition 3.7 and Theorem 3.10 to understand when representations  $V_{k, \omega}$  have fixed vectors when restricted to various reductive subgroups  $\mathbf{G}'$ . Moreover, it suffices to consider these branching problems over the reals. We consider this problem when  $\mathbf{G}'$  is a  $Spin_7$  subgroup, a  $G_2$  subgroup, and an  $SU_3$  subgroup obtained as before as the stabilizer of a triple  $(\alpha, \beta, \gamma)$  of octonions.

All groups in this section will be compact simply-connected real Lie groups; we simply write  $Spin_8$ ,  $Spin_7$ ,  $G_2$ , and  $SU_3$  for these groups. The irreducible representations of  $Spin_8$  are the  $V_{k, \omega}$  we have already discussed, with  $k \geq 0$ , and  $\omega = (\omega_1, \omega_2, \omega_3)$ . The irreducible representations of  $Spin_7$  are indexed by triples  $(m, n, r)$  of non-negative integers in the following way: the triple  $(1, 0, 0)$  corresponds to the 7-dimensional representation, the triple  $(0, 1, 0)$  corresponds to the 21-dimensional adjoint representation, and the triple  $(0, 0, 1)$  corresponds to the 8-dimensional spin representation. General triples  $(m, n, r)$  correspond to representations whose highest weight is the appropriate linear combination of the fundamental weights for the aforementioned representations. Write  $U_{m, n, r}$  for this irreducible representation.



The irreducible representations of  $G_2$  correspond to pairs  $(p, q)$ , where the first coordinate corresponds to the 7-dimensional representation, and the second coordinate corresponds to the 14-dimensional adjoint representation. Write  $T_{p,q}$  for the irreducible representation parametrized as such.

**Proposition 3.11.** *Suppose that  $\omega = 0 = (0, 0, 0)$  and  $k > 0$ . Then  $\text{Res } \downarrow_{G_2} (V_{k,\omega})$  does not contain the trivial representation. However  $\text{Res } \downarrow_{SU_3} (V_{k,\omega})$  does contain the trivial representation.*

*Proof.* Choose any  $Spin_7$  subgroup of  $Spin_8$  containing  $G_2$  (there are three conjugacy classes of such  $Spin_7$  subgroups, by Varadarajan [36]). Then by a well-known branching law, the representation  $V_{k,0}$  restricts as follows:

$$\text{Res } \downarrow_{Spin_7} V_{k,0} = \bigoplus_{m+n=k} U_{m,n,0}.$$

By the branching formula in Theorem 3.4 of [33], the representation  $\text{Res } \downarrow_{G_2} U_{m,n,0}$  contains no trivial representation if  $m + n > 0$ . Hence we see that  $\text{Res } \downarrow_{G_2} V_{k,0}$  contains no trivial representation.

Now by a branching formula in [34], we see that  $\text{Res } \downarrow_{SU_3} T_{p,q}$  contains the trivial representation if and only if  $q = 0$ . Using McGovern's formula in Theorem 3.4 of [33] again, it follows that  $\text{Res } \downarrow_{SU_3} U_{m,n,0}$  always contains the trivial representation. Hence  $\text{Res } \downarrow_{SU_3} V_{k,\omega}$  always contains the trivial representation for  $\omega = (0, 0, 0)$ .  $\square$

If  $V_{k,\omega}$  is centrifugal, i.e.,  $k = 0$ , then it follows that  $\text{Res } \downarrow_{Spin_7} V_{k,\omega}$  contains the trivial representation of  $Spin_7$  for some  $Spin_7$  subgroup of  $Spin_8$ . Hence we have:

**Proposition 3.12.** *If  $k = 0$ , then  $\text{Res } \downarrow_{G_2} V_{k,\omega}$  contains the trivial representation.*

**Corollary 3.13.** *Suppose that  $f$  is an algebraic modular form of level 1 and weight  $(k, \omega)$ . If  $k = 0$ , then  $f$  has non-vanishing period along a  $G_2$  subgroup. If  $\omega = (0, 0, 0)$  and  $k > 0$ , then the periods of  $f$  along  $G_2$  subgroups vanish, but  $f$  has a non-vanishing period along an  $SU_3$  subgroup.*

*Proof.* This follows from the last two propositions, together with Theorem 3.10.  $\square$

#### 4. LOCAL THETA CORRESPONDENCES

In this section, we study a theta correspondence which can be used to construct modular forms on  $\underline{\mathbf{G}}_s$  and  $\underline{\mathbf{SL}}_2^3$  from those on  $\underline{\mathbf{G}}_c$ . We use the dual pairs:

$$\underline{\mathbf{G}}_s \times_{\nu} \underline{\mathbf{G}}_c \hookrightarrow \underline{\mathbf{E}}_{8,4},$$

$$\underline{\mathbf{SL}}_2^3 \times_{\nu} \underline{\mathbf{G}}_c \hookrightarrow \underline{\mathbf{E}}_{7,3}.$$

As  $\mathbf{G}_s$  and  $\mathbf{G}_c$  are inner forms of each other, their Langlands dual groups coincide, and we fix an identification:

$${}^L\mathbf{G}_s \cong {}^L\mathbf{G}_c.$$

**4.1. Real Correspondence.** We describe local results over  $\mathbb{R}$  in this section. In the work of Gross and Wallach [21], they construct the minimal representation  $\Pi_{\mathbb{R}}$  of the quaternionic group  $\mathbf{E}_{8,4}(\mathbb{R})$  by continuation of quaternionic discrete series. Recall that irreducible representations of  $\mathbf{G}_c(\mathbb{R})$  are indexed by pairs  $(k, \omega)$ ,  $\omega = (\omega_1, \omega_2, \omega_3)$ , with  $k, \omega_i$  non-negative. Let  $V_{k,\omega}$  be the irreducible representation associated to the pair  $(k, \omega)$ . If  $k = 0$ , we call  $V_{k,\omega}$  a *centrifugal* representation

– these will play a special role in the theta correspondence. Also, recall that for  $(k, \omega)$  even,  $\pi_{k, \omega}$  is a quaternionic discrete series representation for  $k \geq 9$ .

The following is one of the main results of the paper of Loke [30]:

**Theorem 4.1.** *Upon restriction to the dual pair  $\mathbf{G}_s(\mathbb{R}) \times_\nu \mathbf{G}_c(\mathbb{R})$ , the representation  $\Pi_{\mathbb{R}}$  decomposes as a direct sum over the set of irreducible representations of  $\mathbf{G}_c(\mathbb{R})$ :*

$$\text{Res}(\Pi_{\mathbb{R}}) = \bigoplus_{(k, \omega)} \Theta(V_{k, \omega}) \boxtimes V_{k, \omega},$$

where each  $\Theta(V_{k, \omega})$  is isotypic, consisting of the single quaternionic discrete series  $\pi_{|\omega|+2k+10, \omega}$  with finite multiplicity equal to  $k+1$ , where  $|\omega| = \omega_1 + \omega_2 + \omega_3$ .

In particular, the pairing above is perfect for centrifugal representations, and the trivial representation of  $\mathbf{G}_c(\mathbb{R})$  is paired with the quaternionic discrete series of weight  $k = 10$ , and  $\omega = 0$ . Finally, as remarked in 1.7 of [30], this pairing of representations is the same as that predicted by Langlands functoriality.

Now let  $\Pi'_{\mathbb{R}}$  be the minimal representation of  $\mathbf{E}_{7,3}(\mathbb{R})$ . Let  $D_k$  denote the holomorphic discrete series representation of  $\mathbf{SL}_2(\mathbb{R})$  whose minimal  $K$ -type corresponds to the positive integer  $k$ . A local result due to Gross and Savin, Proposition 3.3 of [20], is:

**Theorem 4.2.** *Upon restriction to the dual pair  $\mathbf{SL}_2(\mathbb{R})^3 \times_\nu \mathbf{G}_c(\mathbb{R})$ , the representation  $\Pi'_{\mathbb{R}}$  decomposes as a direct sum over the set of irreducible representations  $V_{k, \omega}$  of  $\mathbf{G}_c(\mathbb{R})$  with  $k = 0$ :*

$$\text{Res}(\Pi'_{\mathbb{R}}) = \bigoplus_{\omega} \Theta'(V_{0, \omega}) \boxtimes V_{0, \omega},$$

where each  $\Theta'(V_{0, \omega})$  is given by:

$$\Theta'(V_{0, \omega}) = D_{4+|\omega|-\omega_1} \boxtimes D_{4+|\omega|-\omega_2} \boxtimes D_{4+|\omega|-\omega_3}.$$

While  $\Theta'$  is not functorial in the usual sense, it might be referred to as “backwards functoriality”. Note that only centrifugal representations occur in the theta correspondence for  $\Pi'_{\mathbb{R}}$ .

The remainder of this section will be devoted to the  $p$ -adic versions of the above theorems, in the spherical tempered case.

**4.2. Split  $D_4$  geometry.** We begin the process of proving the  $p$ -adic theta correspondence, using methods found in the work of Magaard and Savin [32]. The first step will be geometric. The geometry of “amber spaces” in the 27-dimensional module for  $E_6$ , discussed in [1] and used in [32], will be replaced by a suitable geometry for  $D_4$  originated by Tits in [35].

Recall that the algebraic groups  $\mathbf{G}_c$  and  $\mathbf{G}_s$  are isomorphic and split over  $\mathbb{Q}_p$ . Thus we write  $G$  for the  $\mathbb{Q}_p$ -points of these groups, working consistently over  $\mathbb{Q}_p$  in this section. Following a tradition of abuse, we call subgroups of  $G$  parabolic subgroups, tori, etc..., if they are the  $\mathbb{Q}_p$  points of such algebraic subgroups of  $\mathbf{G}$ .

Fixing a pinning  $T \subset B \subset G$  of  $G$ , there are 16 standard parabolic subgroups of  $G$ , corresponding to subsets of the set of simple roots. The geometric interpretation of these parabolic subgroups, as stabilizers of certain flags, can be expressed in the language of [35]. Of course, viewing  $G$  as a classical group, the parabolic subgroups have an interpretation as stabilizers of isotropic flags in (any of) the 8-dimensional algebraic representations of  $G$ . It is more canonical to view parabolic subgroups

as stabilizers of flags in the full 24-dimensional isotopy representation of  $G$ , as this does not single out a single 8-dimensional representation.

Thus we prefer, and in fact are required in what comes later, to view  $G$  as an exceptional group. In this section, we write  $\mathbb{O}_p$  for the split (and only) octonion algebra over  $\mathbb{Q}_p$ . With this in mind, we define:

**Definition 4.3.** Let  $i, j, k$  be non-negative integers. Let  $\mathcal{F}l_{i,j,k}$  denote the set of triples  $(A, B, C)$  where  $A, B, C$  are  $\mathbb{Q}_p$ -subspaces of  $\mathbb{O}_p$  of dimensions  $i, j, k$  respectively, satisfying  $N(A) = N(B) = N(C) = 0$  and  $AB = BC = CA = 0$ . In other words the norm of any octonion in  $A$  is 0, and the product of any octonion in  $A$  with any octonion in  $B$  is 0, etc...

A number of remarks are in order, following work in [35].

- The set of singular lines in  $\mathbb{O}_p$ , e.g.  $\mathcal{F}l_{1,0,0}$ , is a 6-dimensional quadric hypersurface in  $\mathbb{P}^7$ .
- The set of 3-dimensional hyperplanes in  $\mathcal{F}l_{1,0,0}$  (abrégé in [35]) come in two families, which may be identified with  $\mathcal{F}l_{0,1,0}$  and  $\mathcal{F}l_{0,0,1}$ . The symmetry between these two families and the points of the original quadric is known as triality.
- Incidence among points of  $\mathcal{F}l_{1,0,0}$  and points of  $\mathcal{F}l_{0,1,0}$  can be described as a point belonging to a 3-dimensional hyperplane, or as vanishing of octonionic multiplication. This is described at the end of [35].
- Every two-dimensional subspace  $A$  of  $\mathbb{O}_p$  satisfying  $N(A) = 0$  determines two other such subspaces, given by  $B = \{b: Ab = 0\}$  and  $C = \{c: cA = 0\}$ . Thus  $\mathcal{F}l_{2,0,0}$  is the same as  $\mathcal{F}l_{2,2,0}$  and  $\mathcal{F}l_{2,2,2}$ .

The generalized flag varieties for  $G$  can now be described via  $\mathcal{F}l_{i,j,k}$  for  $i, j, k$  equal to 0, 1, 2.  $G$  acts on  $\mathcal{F}l_{i,j,k}$  via the isotopy representation on  $\mathbb{O}_p^3$ ; we list the parabolic subgroups  $P_{i,j,k}$  stabilizing a point on  $\mathcal{F}l_{i,j,k}$  for all  $i, j, k$ . Note that the standard parabolic subgroups of  $G$  are determined by subsets of the set  $\{\alpha_0, \dots, \alpha_3\}$  of simple roots.

$i, j, k$	Subset of $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$	Dimension of $\mathcal{F}l_{i,j,k}$
0, 0, 0	$\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$	0
1, 0, 0	$\{\alpha_0, \alpha_2, \alpha_3\}$	6
1, 1, 0	$\{\alpha_0, \alpha_3\}$	9
1, 1, 1	$\{\alpha_0\}$	11
2, 1, 1	$\{\alpha_1\}$	11
2, 2, 1	$\{\alpha_1, \alpha_2\}$	10
2, 2, 2	$\{\alpha_1, \alpha_2, \alpha_3\}$	9

**4.3. The minimal representation of  $\mathbf{E}_7(\mathbb{Q}_p)$ .** We write  $E' = \mathbf{E}_{7,3}(\mathbb{Q}_p)$ , noting that  $\mathbf{E}_{7,3}$  is split over  $\mathbb{Q}_p$ . Let  $Q_E = M_E N_E$  denote the (standard) maximal parabolic of  $E'$  with abelian unipotent radical  $N_E$ .  $N_E$  can be identified with the exceptional Jordan algebra  $J_3 = J_3(\mathbb{O}_p)$ . Also, let  $Q = MN$  denote the standard Borel subgroup of  $SL_2^3$ , consisting of triples of upper-triangular matrices;  $M$  is a split torus of rank 3, and  $N$  is unipotent abelian of dimension 3. There is a dual pair embedding  $SL_2^3 \times_\nu G \hookrightarrow E'$  so that  $Q_E \cap (SL_2^3 \times_\nu G) = Q \times_\nu G$ . The three-dimensional unipotent radical  $N$  of  $Q$  is identified with the diagonal elements of the Jordan algebra  $J_3$ . This inclusion yields an orthogonal subspace  $J_3^{\perp N}$  consisting

of elements of the Jordan algebra  $J_3$  with zeroes along the diagonal. Thus  $J_3^{\perp N}$  is identified with the set of triples of octonions  $\mathbb{O}_p^3$ .

The Levi factor  $M_E$  is isomorphic to  $CE_6$ , and acts via the 27-dimensional minuscule representation on  $J_3$ . Let  $\Omega'$  denote the orbit of a highest weight vector in  $J_3$ . Define  $\Omega'^{\perp N} = \Omega' \cap J_3^{\perp N}$ .  $\Omega'$  is precisely the set of rank 1 elements of  $J_3$ , which implies:

**Lemma 4.4.** *The set  $\Omega'^{\perp N}$  is the set of triples  $(\alpha, \beta, \gamma)$  of octonions (not all of which are zero) such that  $\alpha\beta = \beta\gamma = \gamma\alpha = 0$ , and  $N(\alpha) = N(\beta) = N(\gamma) = 0$ .*

We can use this lemma to break  $\Omega'^{\perp N}$  into locally closed pieces, based on the vanishing of  $\alpha$ ,  $\beta$ , or  $\gamma$ . Namely, for  $i, j, k$  equal to 0 or 1, let  $\mathcal{E}_{i,j,k}$  be the fibre bundle over  $\mathcal{F}l_{i,j,k}$  whose fibre over a point  $(A, B, C)$  (a triple of subspaces of  $\mathbb{O}_p$  of dimensions  $i, j, k$ ) is the set of triples  $(\alpha, \beta, \gamma)$  spanning  $(A, B, C)$ . Then we have the decomposition:

$$\Omega'^{\perp N} = \bigsqcup_{i,j,k \in \{0,1\}} \mathcal{E}_{i,j,k}.$$

Not all of the  $i, j, k$  can equal zero, since the point  $(0, 0, 0)$  is not in  $\Omega'^{\perp N}$ .

Let  $\Pi'$  denote the minimal representation of  $E'$ . By Theorem 1.1 of [32], the co-invariants of  $\Pi'$  along the opposite unipotent radical  $\bar{N}$  may be computed:

$$(4.1) \quad 0 \rightarrow C_c^\infty(\Omega'^{\perp N}) \rightarrow \Pi'_{\bar{N}} \rightarrow \Pi'_{\bar{N}_E} \rightarrow 0.$$

Restricting  $\Pi'$  to the dual pair  $SL_2^3 \times_\nu G$ , and taking co-invariants as above yields a representation of  $M \times_\nu G = \mathbb{G}_m^3 \times_\nu G$ . The action of  $\mathbb{G}_m^3 \times_\nu G$  on the terms in the above exact sequence is described by Theorem 1.1 of [32], and we recall this here.

Let  $Isot$  denote the “isotopy” representation of  $G$  on  $\mathbb{O}_p^3$ , i.e., the direct sum of the three inequivalent 8-dimensional representations. Let  $\mathbb{G}_m^3$  also act on  $\mathbb{O}_p^3$  by scaling in the obvious way. We write  $Isot^1$  for the resulting representation of  $\mathbb{G}_m^3 \times_\nu G$  on  $\mathbb{O}_p^3$ . The aforementioned results of Magaard and Savin [32] imply:

**Proposition 4.5.** *The action of  $\mathbb{G}_m^3 \times_\nu G$  on  $C_c^\infty(\Omega'^{\perp N})$  is given by:*

$$[(t, g)f](\alpha, \beta, \gamma) = |t_1 t_2 t_3|^{-4} f(Isot^1(t^{-1}, g^{-1})(\alpha, \beta, \gamma)),$$

where  $t = (t_1, t_2, t_3) \in \mathbb{G}_m^3$  and  $g \in G$ .

The space  $\Pi'_{\bar{N}_E}$  also admits a representation of  $\mathbb{G}_m^3 \times_\nu G$ , also described in Theorem 1.1 of [32]. The Levi component  $M_E$  is isomorphic to  $CE_6$ , and we let  $\Pi''$  denote the minimal representation of  $E_6$ , extended to  $CE_6$  by having the center act trivially, and restricted to the pair  $\mathbb{G}_m^3 \times_\nu G$  in  $CE_6$ . Then we have:

**Proposition 4.6.** *The action of  $\mathbb{G}_m^3 \times_\nu G$  on  $\Pi'_{\bar{N}_E}$  is given by the representation:*

$$\Pi'_{\bar{N}_E} \cong (\Pi'' \otimes |t_1 t_2 t_3|^{-2}) \oplus |t_1 t_2 t_3|^{-4}.$$

**4.4. Tempered spherical representations.** Suppose that  $\tau$  is a tempered spherical representation of  $SL_2^3$ , with regular parameter, so that:

$$\tau = \text{Ind}_Q^{SL_2^3} \chi_1(t_1) \chi_2(t_2) \chi_3(t_3) |t_1 t_2 t_3|^{-1}.$$

The characters  $\chi_i$  must be unitary and unramified.

We also consider irreducible smooth representations  $\pi$  of  $G$ . If  $\pi$  is tempered spherical, there exist unramified characters of  $\mathbb{Q}_p^\times$ ,  $\psi_0, \psi_1, \psi_2, \psi_3$  so that  $\pi$  occurs in  $\text{Ind}_B^G \prod \psi_i \delta_B^{1/2}$ .

**Definition 4.7.** A tempered spherical representation  $\pi$  is called *centrifugal* if it occurs in  $\text{Ind}_B^G \prod \psi_i \delta_B^{1/2}$  with  $\psi_0$  the trivial character.

In other words, centrifugal representations are those whose Satake parameters are in the image of the inclusion of Langlands dual groups  $PGL_2^3 \rightarrow {}^L G$ . For centrifugal representations, the induction from  $T$  naturally factors through the three-step parabolic subgroup  $P_{1,1,1}$ ; the Levi component  $L_{1,1,1}$  of  $P_{1,1,1}$  is the set of quadruples  $(s, t_1, t_2, t_3)$  with  $s \in GL_2$ ,  $t_i \in \mathbb{G}_m$ , and  $\det(s) \cdot t_1 t_2 t_3 = 1$ .

**Proposition 4.8.** Suppose that  $\pi$  is centrifugal, with parameters  $\psi_1, \psi_2, \psi_3$  ( $\psi_0$  trivial by definition). Then  $\pi$  occurs in the induced representation:

$$\text{Ind}_{P_{1,1,1}}^G \psi_1(t_1) \psi_2(t_2) \psi_3(t_3) |t_1 t_2 t_3|^{-3}.$$

*Proof.* This follows from inducing in stages, and an elementary computation of the modular character for  $P_{1,1,1}$ .  $\square$

We refer to the triple  $(\psi_1, \psi_2, \psi_3)$  as the parameter for a tempered spherical centrifugal representation.

Now, let  $\pi$  be any irreducible smooth representation of  $G$ . Applying Frobenius reciprocity, we have:

$$\text{Hom}_{SL_2^3 \times_\nu G}(\Pi', \tau \boxtimes \pi) = \text{Hom}_{\mathbb{G}_m^3 \times_\nu G}(\Pi'_N, (\chi_1(t_1) \chi_2(t_2) \chi_3(t_3) |t_1 t_2 t_3|^{-1}) \boxtimes \pi).$$

Since the characters  $\chi_i$  are unitary, and  $|t_1 t_2 t_3|^{-1}$  is distinct from the two characters  $|t_1 t_2 t_3|^{-2}$  and  $|t_1 t_2 t_3|^{-4}$ , we immediately get from the short exact sequence 4.1:

$$\text{Hom}_{SL_2^3 \times_\nu G}(\Pi', \tau \boxtimes \pi) = \text{Hom}_{\mathbb{G}_m^3 \times_\nu G}(C_c^\infty(\Omega'^{\perp N}), (\chi_1(t_1) \chi_2(t_2) \chi_3(t_3) |t_1 t_2 t_3|^{-1}) \boxtimes \pi).$$

Furthermore, we claim:

**Lemma 4.9.** The only part of  $C_c^\infty(\Omega'^{\perp N})$  that contributes in the above equality is  $C_c^\infty(\mathcal{E}_{1,1,1})$ . That is,

$$\begin{aligned} & \text{Hom}_{\mathbb{G}_m^3 \times_\nu G}(C_c^\infty(\Omega'^{\perp N}), (\chi_1(t_1) \chi_2(t_2) \chi_3(t_3) |t_1 t_2 t_3|^{-1}) \boxtimes \pi) \\ &= \text{Hom}_{\mathbb{G}_m^3 \times_\nu G}(C_c^\infty(\mathcal{E}_{111}), (\chi_1(t_1) \chi_2(t_2) \chi_3(t_3) |t_1 t_2 t_3|^{-1}) \boxtimes \pi). \end{aligned}$$

*Proof.* We look at the possible central characters of  $C_c^\infty(\mathcal{E}_{i,j,k})$  for  $i, j, k \in \{0, 1\}$ , viewed as representations of  $\mathbb{G}_m^3 \times_\nu G$ . The possible characters are tabulated below (up to triality symmetry):

$i, j, k$	Central character
1, 0, 0	$\rho_1(t_1)  t_1 t_2 t_3 ^{-4}$
1, 1, 0	$\rho_1(t_1) \rho_2(t_2)  t_1 t_2 t_3 ^{-4}$
1, 1, 1	$\rho_1(t_1) \rho_2(t_2) \rho_3(t_3)  t_1 t_2 t_3 ^{-4}$

The characters  $\rho_i$  in the above table may be arbitrary smooth characters of  $\mathbb{G}_m$ . These sets are disjoint from the central character  $(\chi_1(t_1) \chi_2(t_2) \chi_3(t_3) |t_1 t_2 t_3|^{-1})$  when  $\chi_i$  are unramified unitary, except when  $i = j = k = 1$ . The lemma follows.  $\square$

Using this lemma, it is not hard to prove a local theta correspondence:

**Theorem 4.10.** *Let  $\tau$  be a tempered spherical representation of  $SL_2^3$ , with regular parameter. Let  $\pi$  be an irreducible spherical representation of  $G$ . Then  $\tau \boxtimes \pi$  occurs as a quotient of the restriction to  $SL_2^3 \times_\nu G$  of  $\Pi'$  if and only if  $\pi$  is a tempered spherical centrifugal representation, whose parameters match those of  $\tau$ .*

*Proof.* From the lemma, we see that  $\tau \boxtimes \pi$  occurs as a quotient of  $\Pi'$  if and only if  $((\chi_1(t_1)\chi_2(t_2)\chi_3(t_3)|t_1t_2t_3|^{-1})) \boxtimes \pi$  occurs as a quotient of  $C_c^\infty(\mathcal{E}_{1,1,1})$ . Now  $\mathcal{E}_{1,1,1}$  is a fibre bundle over  $\mathcal{F}_{1,1,1}$ , with all fibres isomorphic to  $\mathbb{G}_m^3$ , and where  $\mathbb{G}_m^3$  acts fibrewise in the obvious way, and the Levi component of  $P_{1,1,1}$ ,  $L_{1,1,1} \cong \mathbb{G}_m^3 \times SL_2$  acts fibrewise via the obvious representations of  $\mathbb{G}_m^3 \subset L_{1,1,1}$ .

Hence, we arrive at the following description of  $C_c^\infty(\mathcal{E}_{1,1,1})$ :

$$C_c^\infty(\mathcal{E}_{1,1,1}) \cong |t_1t_2t_3|^{-4} \text{Ind}_{\mathbb{G}_m^3 \times_\nu P_{1,1,1}}^{\mathbb{G}_m^3 \times_\nu G} C_c^\infty(\mathbb{G}_m) \boxtimes C_c^\infty(\mathbb{G}_m) \boxtimes C_c^\infty(\mathbb{G}_m).$$

The  $C_c^\infty(\mathbb{G}_m)$  are essentially regular representations. Thus  $\tau \boxtimes \pi$  occurs as a quotient of the restriction of  $\Pi'$  if and only if:

$$\text{Hom}(\text{Ind}_{P_{1,1,1}}^G |t_1t_2t_3|^{-3} \chi_1(t_1)\chi_2(t_2)\chi_3(t_3), \pi) \neq 0.$$

Since we assume that  $\pi$  is spherical, and the above induced representation has a unique spherical constituent, the theorem follows.  $\square$

*Remark 4.11.* It is possible that the above theorem holds for general irreducible smooth representations  $\pi$ . However, this seems to require an analysis of the reducibility of degenerate principal series induced from  $P_{1,1,1}$ . As there are many (52, according to a MAPLE-assisted computation)  $P_{1,1,1}$  double-cosets in  $G$ , this analysis seems cumbersome without any additional insight.

**4.5. The minimal representation of  $\mathbf{E}_8(\mathbb{Q}_p)$ .** The previous work on the minimal representation for  $E' = \mathbf{E}_{7,3}(\mathbb{Q}_p)$  was a good warm-up to the more technical but similar work for  $E = \mathbf{E}_{8,4}(\mathbb{Q}_p)$ . The techniques are essentially the same; however, instead of distinguishing representations by central character, we must instead use strategically chosen elements of the Bernstein center.

Noting that  $\mathbf{E}_{8,4}$  is split over  $\mathbb{Q}_p$ , and let  $P_E = L_E H_E$  denote the Heisenberg parabolic of  $E$  as described for instance in [21]. The unipotent radical  $H_E$  of  $P_E$  has center  $Z$ , and  $F = H_E/Z$  is a 56-dimensional  $\mathbb{Q}_p$  vector space. Let  $P = LH$  denote the Heisenberg parabolic of  $G$ , so that  $P = P_{2,2,2}$  in the previous.  $H$  is 9-dimensional, with one-dimensional center  $Z$ , and  $H/Z$  is the vector space  $C$  of 2 by 2 by 2 cubes over  $\mathbb{Q}_p$ . There is a dual pair embedding  $G \times_\nu G \hookrightarrow E$  so that  $P_E \cap (G \times_\nu G) = P \times_\nu G$ , and the centers  $Z$  of  $H_E$  and  $H$  are identified.

The 56-dimensional space  $F = H_E/Z$  can be viewed as the space of 2 by 2 matrices  $\begin{pmatrix} x & A_+ \\ A_- & y \end{pmatrix}$  where  $x, y \in \mathbb{Q}_p$ , and  $A_\pm$  are contained in the exceptional Jordan algebra  $J_3(\mathbb{O}_p)$ . The inclusion of the 8-dimensional space of cubes  $C$  in  $F$  yields an orthogonal subspace  $F^{\perp C}$ . It is not hard to see that  $F^{\perp C}$  consists of 2 by 2 matrices as above, where  $x = y = 0$ , and all diagonal entries of  $A_\pm$  are 0. Hence elements of  $F^{\perp C}$  are sextuples of octonions:

$$F^{\perp C} = \{(\alpha_\pm, \beta_\pm, \gamma_\pm) \in \mathbb{O}_p^6\}.$$

The Levi factor  $L_E$  of  $P_E$  is isomorphic to  $CE_7$ , and acts via the 56-dimensional minuscule representation on  $F$ . Let  $\Omega$  denote the orbit of a highest weight vector in  $F$ . Define  $\Omega^{\perp C} = \Omega \cap F^{\perp C}$ . An analysis identical to that in Section 7 of [32] yields:

**Lemma 4.12.** *The set  $\Omega^{\perp C}$  is the set of sextuples  $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  of octonions such that if  $A, B, C$  are the spans of  $\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}$  respectively, then  $N(A) = N(B) = N(C) = 0$  and  $AB = BC = CA = 0$ . The sextuple  $\alpha_{\pm} = \beta_{\pm} = \gamma_{\pm} = 0$  is excluded from  $\Omega^{\perp C}$*

From this lemma, we may break up  $\Omega^{\perp C}$  into a finite number of locally closed subsets. For  $i, j, k$  between 0 and 2, let  $\mathcal{E}_{i,j,k}$  be the fibre bundle over  $\mathcal{F}l_{i,j,k}$  whose fibre over a point  $(A, B, C)$  (a triple of subspaces of dimensions  $i, j, k$  in  $\mathbb{O}_p$ ) is the set of sextuples  $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$  spanning  $(A, B, C)$  as in the lemma above. We see immediately that:

$$\Omega^{\perp C} = \bigsqcup_{i,j,k \in \{0,1,2\}} \mathcal{E}_{i,j,k}.$$

Again, not all of  $i, j, k$  can equal 0 in this decomposition.

Let  $\Pi$  denote the minimal representation of  $E$ . Recalling Theorem 6.1 of [32], the co-invariants of  $\Pi$  along the unipotent subgroup  $\bar{Z}$  opposite to  $Z$  may be decomposed:

$$0 \rightarrow C_c^{\infty}(\Omega) \rightarrow \Pi_{\bar{Z}} \rightarrow \Pi_{\bar{H}_E} \rightarrow 0.$$

Furthermore, taking the co-invariants along all of  $\bar{H}$ , we get:

$$(4.2) \quad 0 \rightarrow C_c^{\infty}(\Omega^{\perp C}) \rightarrow \Pi_{\bar{H}} \rightarrow \Pi_{\bar{H}_E} \rightarrow 0.$$

Restricting  $\Pi$  to the dual pair  $G \times_{\nu} G$ , and taking co-invariants shows that the three terms in the above short exact sequence are representations of  $L \times_{\nu} G$ . These representations of  $L \times_{\nu} G$  are described by Theorem 6.1 of [32]. We begin with the action on  $C_c^{\infty}(\Omega^{\perp C})$ .

Recall  $Isot$  is the “isotopy” representation of  $G$  on the 24-dimensional space of triples of octonions  $(\alpha, \beta, \gamma)$ . Let  $St_{\mathbb{O}_p}$  denote the “standard” action of  $SL_2$  on pairs  $(\kappa_+, \kappa_-)$  of octonions, given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \kappa_+ \\ \kappa_- \end{pmatrix} = \begin{pmatrix} a\kappa_+ + b\kappa_- \\ c\kappa_+ + d\kappa_- \end{pmatrix}.$$

Let  $Isot^2$  denote the resulting 48-dimensional representation of  $SL_2^3 \times_{\nu} G$  on sextuples of octonions  $(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})$ . In other words,  $G$  acts on the triples  $(\alpha_+, \beta_+, \gamma_+)$  and  $(\alpha_-, \beta_-, \gamma_-)$  via the isotopy representation, and the three  $SL_2$ ’s act on the three pairs  $\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}$  via the standard representation. Extend  $Isot^2$  to  $L \times_{\nu} G$  by letting the central  $\mathbb{G}_m$  in  $L$  act by uniformly scaling  $\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}$ . Then Theorem 6.1 of [32] implies:

**Proposition 4.13.** *The action of  $L \times_{\nu} G$  on  $C_c^{\infty}(\Omega^{\perp C})$  is given by:*

$$[(l, g)f](\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm}) = |\det|^{-5} f(Isot^2(l^{-1}, g^{-1})(\alpha_{\pm}, \beta_{\pm}, \gamma_{\pm})),$$

where  $\det$  denotes the determinant character on  $L$ .

The space  $\Pi_{\bar{H}_E}$  also admits a representation of  $L \times_{\nu} G$ , which is described in Theorem 6.1 of [32]. Let  $\Pi'$  denote the minimal representation of  $E_7$ , extended to  $CE_7$  by having the center act trivially, and restricted to the dual pair  $L \times_{\nu} G$  in  $CE_7$ . Again,  $\det$  will denote the determinant character on  $L$ . Then we have:

**Proposition 4.14.** *The action of  $L \times_{\nu} G$  on  $\Pi_{\bar{H}_E}$  is given by the representation:*

$$\Pi_{\bar{H}_E} \cong (\Pi' \otimes |\det|^{-3}) \oplus |\det|^{-5}.$$

**4.6. The Bernstein center for  $L$ .** We follow the methods of Section 4 of Magaard-Savin [32] to pick out certain representations of  $L$ , using certain elements of the Bernstein center of  $L$ . Recalling that  $L$  is the group of triples  $l = (l_1, l_2, l_3)$  of matrices in  $GL_2$  such that  $\det(l_1) = \det(l_2) = \det(l_3)$ , we begin by writing down a basis for the lattices  $X_\bullet(T)$  and  $X^\bullet(T)$ . As a basis for  $X_\bullet(T)$ , we choose:

$$\begin{aligned}\lambda_0(t) &= \left( \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \right), \\ \lambda_1(t) &= \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \\ \lambda_2(t) &= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), \\ \lambda_3(t) &= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right).\end{aligned}$$

We describe a basis for  $X^\bullet(T)$  as follows: if  $l = (l_1, l_2, l_3) \in T$ , then we let  $\chi_0(l)$  denote the common determinant of  $l_1, l_2, l_3$ . If  $l_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}$ , then we define  $\chi_i(l) = a_i$ , for  $i = 1, 2, 3$ . Then the canonical pairing  $X_\bullet(T) \times X^\bullet(T) \rightarrow \mathbb{Z}$  satisfies:

$$\begin{aligned}\langle \lambda_i, \chi_i \rangle &= 1, \text{ for } i = 0, 1, 2, 3, \\ \langle \lambda_i, \chi_j \rangle &= 0, \text{ for } i \neq j.\end{aligned}$$

The component of the Bernstein center acting non-trivially on representations generated by their Iwahori-fixed vectors is isomorphic to:

$$\mathbb{C}[x_0, x_0^{-1}, x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}]^W,$$

where  $W$  is the abelian group of order 8 generated by  $w_1, w_2, w_3$  acting on the  $x_i$  by:

$$\begin{aligned}w_i x_i &= x_i^{-1} x_0 \text{ for } i = 1, 2, 3, \\ w_i x_0 &= x_0 \text{ for } i = 1, 2, 3, \\ w_i x_j &= x_j \text{ for } i \neq j, j \neq 0.\end{aligned}$$

Here the variables  $x_i$  are identified with the cocharacters  $\lambda_i$ , but we use multiplicative notation for the variables  $x_i$ , rather than additive notation for  $\lambda_i$ .

If  $E$  is a subquotient of an induced representation (from  $T$  to  $L$ ) with parameter  $\chi = (\chi_0, \dots, \chi_3)$ , i.e., induced from the character  $\chi$  extended to the Borel subgroup, then we have:

$$\begin{aligned}x_i + x_0 x_i^{-1}|_E &= \chi_i(p) + \chi_0(p) \chi_i^{-1}(p), \\ x_0|_E &= \chi_0(p).\end{aligned}$$

If  $\tau$  is a tempered spherical representation of  $L$ , then the parameter of  $\tau$  has the form

$$\chi = (\chi_0|\cdot|^{-3/2}, \chi_1|\cdot|, \chi_2|\cdot|, \chi_3|\cdot|),$$

with all  $\chi_i$  unitary characters.



Define the following elements of the Bernstein center of  $L$ :

$$\begin{aligned} T_i &= x_i + x_0 x_i^{-1} \text{ for } i = 1, 2, 3, \\ T_0 &= x_0. \end{aligned}$$

Then the  $T_i$  act on the tempered  $\tau$  above by:

$$\begin{aligned} T_i|_\tau &= \chi_i(p)p^2 + \chi_0(p)\chi_i(p)^{-1}p^{-5/2}, \text{ for } i = 1, 2, 3, \\ T_0|_\tau &= \chi_0(p)p^{-3/2}. \end{aligned}$$

**4.7. Tempered spherical representations.** Suppose that  $\pi$  is a irreducible tempered spherical representation of  $G$  with regular parameter. Then there exists a tempered spherical representation  $\tau$  of  $L$  so that:

$$\pi = \text{Ind}_P^G (\tau \otimes |\det|^{-5}).$$

Let  $\pi'$  be any other smooth representation of  $G$ . Then by Frobenius reciprocity, we have:

$$\text{Hom}(\Pi, \pi \boxtimes \pi') = \text{Hom}(\Pi_{\bar{H}}, (\tau \otimes |\det|^{-5}) \boxtimes \pi').$$

Since  $\tau$  is tempered, it has unitary central character. Thus, there are no non-zero homomorphisms, or non-trivial extensions, from  $\Pi' \otimes |\det|^{-3}$  to  $(\tau \otimes |\det|^{-5}) \boxtimes \pi'$ , since the central characters are disjoint.

We consider when the parameter for  $\pi$  is regular, so that  $\pi$  occurs as an irreducible induced representation.

**Proposition 4.15.** *Suppose that  $\pi$  is a irreducible tempered spherical representation of  $G$ . If we are given  $\pi$  as an induced representation:*

$$\pi = \text{Ind}_P^G (\tau \otimes |\det|^{-5}),$$

*then we have:*

$$\text{Hom}(\Pi, \pi \boxtimes \pi') = \text{Hom}(C_c^\infty(\Omega^{\perp C}), (\tau \otimes |\det|^{-5}) \boxtimes \pi').$$

*Proof.* This is immediate from the short exact sequence 4.2 and Frobenius reciprocity. Note that tempered representations are disjoint from the trivial representation, so that  $\text{Hom}(\pi, 1) = \text{Ext}(\pi, 1) = 0$ .  $\square$

Finally, for  $\pi$  induced from  $\tau$  as before,  $\tau$  must be tempered spherical, and the parameter of  $\tau$  has the form

$$\chi = (\chi_0|\cdot|^{-3/2}, \chi_1|\cdot|, \chi_2|\cdot|, \chi_3|\cdot|),$$

with all  $\chi_i$  unitary characters.

The action of the elements  $p^5 T_i$  on  $\tau \otimes |\det|^{-5}$  is given by:

$$\begin{aligned} p^5 T_i &= \chi_i(p)p^2 + \chi_0(p)\chi_i(p)^{-1}p^{-5/2}, \\ p^5 T_0 &= \chi_0(p)p^{-3/2}. \end{aligned}$$

**4.8. The p-adic correspondence.** The action of the elements  $T_i$  on  $C_c^\infty(\mathcal{E}_{i,j,k})$  can be explicitly computed, since  $L$  acts fibrewise on the bundle  $\mathcal{E}_{i,j,k}$  over  $\mathcal{F}l_{i,j,k}$ . If  $\tau'$  is a spherical representation of  $L$  occurring as a subquotient of  $C_c^\infty(\mathcal{E}_{i,j,k})$ , we denote its parameters by  $(\rho_0, \dots, \rho_3)$ . By knowing the possible parameters for  $\tau'$ , we tabulate the possible non-zero eigenvalues of  $T_i$  on subquotients of  $C_c^\infty(\mathcal{E}_{i,j,k})$  below, normalizing the  $T_i$  by factors of  $p^5$ :

$i, j, k$	Eigenvalues of $p^5 T_0$	Eigenvalues of $p^5 T_1, p^5 T_2, p^5 T_3$
1, 0, 0	$\rho_0(p)$	$\rho_0(p) + 1, 1 + \rho_0(p), 1 + \rho_0(p)$
1, 1, 0	$\rho_0(p)$	$\rho_0(p) + 1, \rho_0(p) + 1, 1 + \rho_0(p)$
1, 1, 1	$\rho_0(p)$	$\rho_0(p) + 1, \rho_0(p) + 1, \rho_0(p) + 1$
2, 1, 1	$\rho_0(p)$	$\rho_1(p) + \frac{\rho_0}{\rho_1}(p), \rho_0(p) + 1, \rho_0(p) + 1$
2, 2, 1	$\rho_0(p)$	$\rho_1(p) + \frac{\rho_0}{\rho_1}(p), \rho_2(p) + \frac{\rho_0}{\rho_2}(p), \rho_0(p) + 1$
2, 2, 2	$\rho_0(p)$	$\rho_1(p) + \frac{\rho_0}{\rho_1}(p), \rho_2(p) + \frac{\rho_0}{\rho_2}(p), \rho_3(p) + \frac{\rho_0}{\rho_3}(p)$

Looking at the above table, the part of every representation  $C_c^\infty(\mathcal{E}_{i,j,k})$  generated by Iwahori-fixed vectors is an eigenspace for the operator  $p^5(T_3 - T_0)$  (or a suitable variation under triality symmetry) of eigenvalue equal to 1, except when  $i = j = k = 2$ . On the other hand, if  $\tau$  is an irreducible tempered spherical representation as before, then the eigenvalue of  $p^5(T_3 - T_0)$  equals:

$$v(\tau) = \chi_3(p)p^2 - \chi_0(p)p^{-3/2} + \chi_0(p)\chi_3(p)^{-1}p^{-5/2}.$$

Since the  $\chi_i$  are unitary, an elementary estimate yields:

$$|v(\tau)| \geq p^2 - p^{-3/2} - p^{-5/2} \geq p^2 - 2 \geq 2.$$

Thus  $v(\tau)$  cannot equal 1. Lemma 2.5 of [32] now yields:

**Lemma 4.16.** *Let  $\tau$  be an irreducible tempered spherical representation. Then  $\tau \otimes |\det|^{-5}$  occurs as a quotient of  $C_c^\infty(\Omega^{\perp C})$  if and only if it occurs as a quotient of  $C_c^\infty(\mathcal{E}_{2,2,2})$ .*

It is now an easy step to get to the following:

**Theorem 4.17.** *Suppose that  $\pi$  is an irreducible tempered spherical representation of  $G$ , whose parameter  $(\chi_0, \dots, \chi_3)$  is regular so that  $\text{Ind}_B^G \chi$  is irreducible. If  $\pi'$  is any irreducible smooth representation of  $G$ , then  $\pi \boxtimes \pi'$  occurs as a quotient of the restriction to  $G \times_\nu G$  of the minimal representation  $\Pi$  if and only if  $\pi \cong \pi'$ .*

*Proof.* By Proposition 4.15, we have

$$\text{Hom}(\Pi, \pi \boxtimes \pi') = \text{Hom}(C_c^\infty(\Omega^{\perp C}), (\tau \otimes |\det|^{-5}) \boxtimes \pi').$$

From the last lemma, this yields:

$$\text{Hom}(\Pi, \pi \boxtimes \pi') = \text{Hom}(C_c^\infty(\mathcal{E}_{2,2,2}), (\tau \otimes |\det|^{-5}) \boxtimes \pi').$$

From Proposition 4.13, we know that as a representation of  $L \times_\nu G$ ,

$$C_c^\infty(\mathcal{E}_{2,2,2}) \cong |\det|^{-5} \otimes \text{Ind}_{L \times_\nu P_{2,2,2}}^{L \times_\nu G} C_c^\infty(GL_2)^{\boxtimes 3}.$$

The representation  $C_c^\infty(GL_2)^{\boxtimes 3}$  is essentially a regular representation of  $L \times_\nu L$ , and so every irreducible representation of  $L \times_\nu L$  occurring as a quotient has the form  $\tau \boxtimes \tau$  for some irreducible smooth  $\tau$ , and all such  $\tau \boxtimes \tau$  occur. Thus by Frobenius reciprocity again, we have:

$$\text{Hom}(\Pi, \pi \boxtimes \pi') \neq 0 \text{ iff } \text{Hom}(\tau \boxtimes \tau, \tau \boxtimes (\pi')_H) \neq 0.$$

By a result, attributed to Bernstein, proven by Bushnell in Theorem 3 of [6], we have:

$$\mathrm{Hom}(\tau, (\pi')_H) = \mathrm{Hom}(\mathrm{Ind}_P^G \tau, \pi').$$

By the irreducibility of  $\mathrm{Ind}_P^G \tau = \pi$ , we have

$$\mathrm{Hom}(\tau, (\pi')_H) = \mathrm{Hom}(\pi, \pi').$$

Finally, we get:

$$\mathrm{Hom}(\Pi, \pi \boxtimes \pi') \neq 0 \text{ iff } \mathrm{Hom}(\tau \boxtimes \pi, \tau \boxtimes \pi') = \mathrm{Hom}(\pi, \pi') \neq 0,$$

and the theorem follows.  $\square$

It seems likely that some of the assumptions in this local theta correspondence could be removed with more technical work. The assumption of regularity is necessary, since the  $R$ -group can be non-trivial for  $G$  by a result of Keys [25]. However, G. Savin has mentioned that working with the adjoint form would eliminate the  $R$ -group, and perhaps the need for the regularity assumption with it. The spherical assumption could also likely be weakened, since in the regular representation of  $L$  used above, various cuspidal representations occur paired with themselves. We leave these details however, until a time when they might be necessary.

## 5. GLOBAL THETA CORRESPONDENCE

In this section, we study global theta correspondences for the same groups we studied locally in the last section. Namely, we hope to lift modular forms on  $\mathbf{G}_c$  to modular forms on  $\mathbf{G}_s$ , and to holomorphic modular forms on  $\mathbf{SL}_2^3$ .

**5.1. The exceptional Jordan algebra.** Recall that  $\Omega_c$  is the Coxeter's ring of integral octonions. From  $\Omega_c$ , it is possible to define the exceptional Jordan algebra  $J_3$  over  $\mathbb{Z}$  (more precisely, the Jordan composition is defined over  $\mathbb{Z}[1/2]$ ). Let  $\mathbf{J}_3$  be the scheme over  $\mathbb{Z}$  underlying the rank 27  $\mathbb{Z}$ -lattice of 3 by 3 Hermitian symmetric matrices over  $\Omega_c$ . An element of  $\mathbf{J}_3(\mathbb{Z})$  can be written in the form:

$$A = \begin{pmatrix} a & \gamma & \bar{\beta} \\ \bar{\gamma} & b & \alpha \\ \beta & \bar{\alpha} & c \end{pmatrix},$$

where  $a, b, c \in \mathbb{Z}$  and  $\alpha, \beta, \gamma \in \Omega_c$ .  $\mathbf{J}_3$  naturally has additional structures. Following [10], we define the cubic form  $\mathrm{Det}(A)$  by:

$$\mathrm{Det}(A) = abc + \mathrm{Tr}(\alpha\beta\gamma) - a \cdot N(\alpha) - b \cdot N(\beta) - c \cdot N(\gamma).$$

We define the adjoint matrix by:

$$A^\sharp = \begin{pmatrix} bc - N(\alpha) & \bar{\beta}\bar{\alpha} - c\gamma & \gamma\alpha - b\bar{\beta} \\ \alpha\beta - c\bar{\gamma} & ca - N(\beta) & \bar{\gamma}\bar{\beta} - a\alpha \\ \bar{\alpha}\bar{\gamma} - b\beta & \beta\gamma - a\bar{\alpha} & ab - N(\gamma) \end{pmatrix}.$$

If  $R$  is a ring, then an element  $A \in \mathbf{J}_3(R)$  is said to have rank 1 if  $A \neq 0$ , but its adjoint  $A^\sharp = 0$ . There are no trace 0 rank 1 elements of  $\mathbf{J}_3(\mathbb{R})$ .

In [12], Freudenthal describes the 56-dimensional representation of  $\mathbf{E}_7(\mathbb{R})$  from  $J_3$ ; following his construction, we define  $\mathbf{F}$  to be the group scheme over  $\mathbb{Z}$  underlying the rank 56  $\mathbb{Z}$ -lattice of 2 by 2 matrices of the form:

$$\phi = \begin{pmatrix} x & A_+ \\ A_- & y \end{pmatrix},$$

where  $x, y \in \mathbb{Z}$  and  $A_+, A_- \in \mathbf{J}_3(\mathbb{Z})$ .

**5.2. Automorphic theta modules.** The automorphic realization of the global minimal representation  $\Pi'$  of  $\mathbf{E}_{7,3}(\mathbb{A})$  follows from the work of Kim in [26]. Thus we have a map:

$$\Theta': \bigotimes_v \Pi'_v \rightarrow L^2(\mathbf{E}_{7,3}(\mathbb{Q}) \backslash \mathbf{E}_{7,3}(\mathbb{A})).$$

The representations  $\Pi'_v$  are the minimal representations of  $\mathbf{E}_{7,3}(\mathbb{Q}_v)$  for every place  $v$  of  $\mathbb{Q}$ .

The work of Gan [13] gives an automorphic realization of the global minimal representation of  $\mathbf{E}_{8,4}$ .

$$\Theta: \bigotimes_v \Pi_v \rightarrow L^2(\mathbf{E}_{8,4}(\mathbb{Q}) \backslash \mathbf{E}_{8,4}(\mathbb{A})).$$

The representations  $\Pi_v$  are the minimal representations of  $\mathbf{E}_{8,4}(\mathbb{Q}_v)$  for every place  $v$  of  $\mathbb{Q}$ .

The global minimal representations are spherical at every finite place (by [13] for  $\mathbf{E}_{8,4}$ ), and the tensor product is taken with respect to suitably normalized spherical vectors. The minimal  $K$ -type of the minimal representation of  $\mathbf{E}_{7,3}(\mathbb{R})$  is one-dimensional, so in this case there is a natural way to choose a vector at the real place as well, up to scaling. Both global minimal representations arise as quotients of globally parabolically induced representations from characters; thus there are natural *normalized* spherical vectors  $t_p, t'_p$  for all finite primes  $p$ , for  $\Pi_p, \Pi'_p$  respectively. A vector  $t \in \Pi$  (or  $t' \in \Pi'$ ) is said to be standard if there is a decomposition  $t = \bigotimes_v t_v$  (or  $t' = \bigotimes_v t'_v$ ), where  $t_v$  (or  $t'_v$ ) is the normalized spherical vector for almost all  $v$ .

Fix  $t$  a standard section of  $\Pi$ . Let  $t'$  denote the global normalized vector, spherical at all finite places, in  $\Pi'$  considered by Kim [26]. Let  $\theta = \Theta(t)$ , and  $\theta' = \Theta'(t')$ . Thus  $\theta$  is an automorphic form on  $\mathbf{E}_{8,4}$  and  $\theta'$  is an automorphic form on  $\mathbf{E}_{7,3}$ . We consider the Fourier expansion of  $\theta$  and  $\theta'$  along the following parabolic subgroups: first, let  $\mathbf{Q}_E = \mathbf{M}_E \mathbf{N}_E$  denote the (standard) maximal parabolic subgroup of  $\mathbf{E}_{7,3}$  with abelian unipotent radical. The unipotent radical  $\mathbf{N}_E$  can naturally be identified with the exceptional Jordan algebra over  $\mathbb{Q}$ :  $\mathbf{N}_E \cong \mathbf{J}_3$ . Second, let  $\mathbf{P}_E = \mathbf{L}_E \mathbf{H}_E$  denote the Heisenberg parabolic of  $\mathbf{E}_{8,4}$ . The derived subgroup of  $\mathbf{L}_E$  is the group  $\mathbf{E}_{7,3}$ . The unipotent radical  $\mathbf{H}_E$  is 57-dimensional, with 1 dimensional center  $\mathbf{Z}$ . The 56-dimensional quotient can be identified with the Freudenthal space  $\mathbf{F}$  described before; the action of the derived subgroup of  $\mathbf{L}_E$  is the minuscule 56-dimensional representation discussed in [12].

Fix highest weight vectors  $w'$  in  $\mathbf{J}_3(\mathbb{Q})$  and  $w$  in  $\mathbf{F}(\mathbb{Q})$  for the action of the derived subgroups of  $\mathbf{M}_E(\mathbb{Q})$  and  $\mathbf{L}_E(\mathbb{Q})$  respectively. Let  $\Omega'$  and  $\Omega$  denote the orbits of these highest weight vectors. The orbit  $\Omega'$  consists precisely of rank 1 elements of  $\mathbf{J}_3(\mathbb{Q})$ . The orbit  $\Omega$  is more difficult to describe, but can be found in [32], Lemma 7.5.

We naturally identify  $\mathbf{F}(\mathbb{Q})$  with the set of characters of  $\mathbf{H}_E(\mathbb{A})$  which are trivial on  $\mathbf{H}_E(\mathbb{Q})$ . Also, we identify  $\mathbf{J}_3(\mathbb{Q})$  with the set of characters of  $\mathbf{J}_3(\mathbb{A})$  which are trivial on  $\mathbf{J}_3(\mathbb{Q})$ .

For a character  $\phi \in \mathbf{J}_3(\mathbb{Q})$ , the Fourier coefficient of  $\theta'$  is defined by:

$$\theta'_\phi(g) = \oint_{\mathbf{J}_3} \theta'(ng) \overline{\phi(n)} dn.$$

From the arguments in Section 5, Subsection 3, of [20], we have:

**Proposition 5.1.** *If  $\phi$  is non-trivial, and  $\theta'$  is non-zero, then  $\theta'_\phi$  is non-zero only if  $\phi$  lies in  $\Omega'$  and equivalently has rank 1. If  $\mathbf{M}_\phi$  denotes the stabilizer of  $\phi$  in the Levi component  $\mathbf{M}_E = \mathbf{E}_{6,2}$ , then  $\theta'_\phi(cg) = \theta'_\phi(g)$  for all  $c \in \mathbf{M}_\phi(\mathbb{A})$ .*

When  $t' = \bigotimes t'_v$  is the normalized spherical vector of  $\Pi'$  at all finite places, and  $t_\infty$  a well-chosen vector in the one-dimensional minimal  $K$ -type, Kim gives more precise information on the Fourier coefficients of  $\theta'$  in [26]:

**Proposition 5.2.** *The constant term of  $\theta'$  is 1, and the non-constant Fourier coefficients are given by non-zero constants (i.e., constant functions on  $\mathbf{E}_{6,2}$ ) for all  $\phi \in \mathbf{J}_3(\mathbb{Z})$ :*

$$a_\phi = 240 \sum_{d|c(\phi)} d^3,$$

where  $c(\phi)$  is the largest integer such that  $c(\phi)^{-1}\phi$  is in  $\mathbf{J}_3(\mathbb{Z})$ .

Now we consider the Fourier expansion of  $\theta$  along the Heisenberg parabolic  $\mathbf{H}_E$ , following Section 6 of [13]. The  $\mathbf{Z}$ -constant term of  $\theta$  is defined by:

$$\theta_{\mathbf{Z}}(g) = \oint_{\mathbf{Z}} \theta(zg) dz.$$

Suppose  $\phi \in \mathbf{F}(\mathbb{Q})$  is viewed as a character of  $\mathbf{H}_E(\mathbb{A})$  trivial on  $\mathbf{H}_E(\mathbb{Q})$ . Then the  $\phi$ -Fourier coefficient of  $\theta$  is defined by

$$\theta_\phi(g) = \oint_{\mathbf{F}} \theta_{\mathbf{Z}}(ng) \overline{\phi(n)} dn.$$

From [13], we know:

**Proposition 5.3.** *If  $\phi$  is non-trivial, and  $\theta$  is non-zero, then  $\theta_\phi$  is non-zero only if  $\phi$  is in the rational orbit  $\Omega$  of a highest weight vector under the action of  $\mathbf{L}_E(\mathbb{Q})$ .*

Furthermore, the constant term, when  $\phi = 0$ , is given by:

**Proposition 5.4.** *The constant term of  $\theta$  along  $\mathbf{H}_E$  is an automorphic form on the derived subgroup of  $\mathbf{L}_E$  (which is isomorphic to  $\mathbf{E}_{7,3}$ );  $\theta_{\mathbf{H}_E}$  is given by  $\theta_{\mathbf{H}_E} = c + \theta'$  where  $c$  is a constant function and  $\theta'$  is contained in the image of the automorphic realization of the minimal representation of  $\mathbf{L}_E(\mathbb{A})$ .*

**5.3. Global theta lift to  $\mathbf{SL}_2^3$ .** We begin by considering the global theta lift from  $\mathbf{G}_c$  to  $\mathbf{SL}_2^3$ . Our methods are the same as those of [13]. Fix a standard section  $t'$  of  $\Pi'$ , and let  $\theta' = \Theta'(t')$  as before. Also fix an automorphic form  $f_c$  on  $\mathbf{G}_c$ . The theta lift of  $f_c$  via  $\theta'$  is defined by

$$\Phi(g) = \int_{\mathbf{G}_c} \theta'(g, g') f_c(g') dg',$$

where  $g \in \mathbf{SL}_2^3(\mathbb{A})$  and  $(g, g')$  is considered as an element of  $\mathbf{E}_{7,3}(\mathbb{A})$  via the dual pair embedding:

$$\mathbf{SL}_2^3 \times_\nu \mathbf{G}_c \hookrightarrow \mathbf{E}_{7,3}.$$

The unipotent radical  $\mathbf{N}$  of the Borel subgroup  $\mathbf{Q}$  of  $\mathbf{SL}_2^3$  is abelian, three-dimensional, and spanned by three subgroups  $\mathbf{N}_I$ ,  $\mathbf{N}_{II}$ ,  $\mathbf{N}_{III}$ . For  $\Phi$  to be cuspidal, we must examine the constant term of  $\Phi$  along each of the three subgroups

$\mathbf{N}_I, \mathbf{N}_{II}, \mathbf{N}_{III}$ . We have:

$$\begin{aligned}\Phi_{\mathbf{N}_I}(g) &= \oint_{\mathbf{N}_I} \oint_{\mathbf{G}_c} \theta'(ng, g') f_c(g') dg' dn, \\ &= \oint_{\mathbf{N}_I} \oint_{\mathbf{G}_c} \sum_{\phi \in \mathbf{J}_3(\mathbb{Q})} \theta'_\phi(ng, g') f_c(g') dg' dn,\end{aligned}$$

where the Fourier coefficients  $\theta'_\phi$  satisfy  $\theta'_\phi(ng, g') = \phi(n)\theta'_\phi(g, g')$  for all  $n \in \mathbf{J}_3(\mathbb{A})$ . Since  $\mathbf{G}_c(\mathbb{Q}) \backslash \mathbf{G}_c(\mathbb{A})$  compact, and the Fourier expansion of  $\theta$  along  $\mathbf{J}_3$  converges absolutely, the integration over  $\mathbf{N}_I(\mathbb{Q}) \backslash \mathbf{N}_I(\mathbb{A})$  may be brought inside. This integration over  $\mathbf{N}_I$  kills most of the Fourier coefficients of  $\theta'$ . The remaining coefficients  $\phi$  are those of the form:

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & \alpha \\ 0 & \bar{\alpha} & c \end{pmatrix},$$

for  $\alpha \in \mathbb{Q}$ ,  $b, c \in \mathbb{Q}$ , and  $N(\alpha) = bc$ . We arrive at:

$$\Phi_{\mathbf{N}_I}(g) = \oint_{\mathbf{G}_c} f_c(g') \sum_{\phi: N(\alpha)=bc} \theta'_\phi(g, g') dg'.$$

If  $\phi$  has the above form, with  $N(\alpha) = bc = 0$ , then the stabilizer in  $\mathbf{G}_c$  of  $\phi$  is all of  $\mathbf{G}_c$ . Otherwise, the stabilizer in  $\mathbf{G}_c$  of a  $\phi$  of the form above is the  $Spin_7$  subgroup  $\mathbf{G}_I(\alpha)$ . The terms where  $N(\alpha) = bc = 0$  may be expressed in terms of a period:

$$\sum_{\phi: \alpha=bc=0} \oint_{\mathbf{G}_c} f_c(g') \theta'_\phi(g, g') dg' = \sum_{\phi: \alpha=bc=0} \theta'_\phi(g, 1) \oint_{\mathbf{G}_c} f_c(g') dg'.$$

As long as  $f_c$  is not  $\mathbf{G}_c$ -distinguished, i.e.,  $f_c$  is orthogonal to constant functions, the above integral vanishes.

For  $N(\alpha) = bc \neq 0$ , the stabilizer in  $\mathbf{G}_c$  of  $\phi$  is the group  $\mathbf{G}_I(\alpha)$ . Moreover,  $\mathbf{G}_c(\mathbb{Q})$  acts transitively on the set of octonions  $\alpha$  of a given non-zero norm  $bc$ . Hence, assuming  $f_c$  is not  $\mathbf{G}_c$ -distinguished, we can unfold the integral:

$$\begin{aligned}\Phi_{\mathbf{N}_I}(g) &= \sum_{\phi: N(\alpha)=bc \neq 0} \oint_{\mathbf{G}_c} f_c(g') \theta'_\phi(g, g') dg' \\ &= \sum_{bc \neq 0} \int_{\mathbf{G}_I(\alpha)(\mathbb{A}) \backslash \mathbf{G}_c(\mathbb{A})} \theta'_{\phi_{bc}}(g, g') \oint_{\mathbf{G}_I(\alpha)} f_c(hg') dh dg' .\end{aligned}$$

Here  $\phi_{bc}$  is a fixed  $\phi$  satisfying  $N(\alpha) = bc \neq 0$ . The inner integral is precisely the period of  $f_c$  along the subgroup  $\mathbf{G}_I(\alpha)$ . Hence we have shown:

**Proposition 5.5.** *If  $f_c$  is not  $\mathbf{G}_I$  (resp.  $\mathbf{G}_{II}, \mathbf{G}_{III}$ ) distinguished, then the theta-lift  $\Phi$  of  $f_c$  is cuspidal along  $\mathbf{N}_I$  (resp.  $\mathbf{N}_{II}, \mathbf{N}_{III}$ ).*

Given the above sufficient condition for cuspidality of  $\Phi$ , we study its non-vanishing. For generic  $a, b, c \in \mathbb{Q}$ , corresponding to a character  $\psi = \psi_{a,b,c}$  of  $\mathbf{N}(\mathbb{A})$ , the  $\psi$ -Fourier coefficient of  $\Phi$  is given by:

$$\Phi_\psi(g) = \oint_{\mathbf{G}_c} f_c(g') \sum_{diag(\phi)=(a,b,c)} \theta'_\phi(g, g') dg',$$

where

$$\phi = \begin{pmatrix} a & \gamma & \bar{\beta} \\ \bar{\gamma} & b & \alpha \\ \beta & \bar{\alpha} & c \end{pmatrix}.$$

If the diagonal entries of  $\phi$  are  $(a, b, c)$  as above, so  $\phi$  restricts to  $\psi$  on  $\mathbf{N}$ , we write  $\text{Res}(\phi) = \psi$ . For  $\phi$  to have rank 1 (these are the only characters for which  $\theta'$  has non-vanishing coefficients), we must have  $N(\alpha) = bc$ ,  $N(\beta) = ca$ ,  $N(\gamma) = ab$  and  $\alpha\beta = c\bar{\gamma}$  as well. Thus the stabilizer of such  $\phi$  in  $\mathbf{G}_c$  is the  $G_2$  subgroup  $\mathbf{G}_2(\alpha, \beta, \gamma)$  of  $\mathbf{G}_c$ .

**Lemma 5.6.** *The group  $\mathbf{G}_c(\mathbb{Q})$  acts transitively on the set of  $\phi$  satisfying  $\text{Res}(\phi) = \psi$  as above.*

*Proof.* Our proof directly follows suggestions of the referee: the  $\mathbf{G}_c(\mathbb{Q})$ -orbits on these  $\phi$  correspond to elements of the kernel of the canonical map:

$$H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbf{G}_2(\alpha, \beta, \gamma)(\bar{\mathbb{Q}})) \rightarrow H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbf{G}_c(\bar{\mathbb{Q}})).$$

Since both of the groups  $\mathbf{G}_2(\alpha, \beta, \gamma)$  and  $\mathbf{G}_c$  are simply-connected, the first Galois cohomology vanishes over every finite place. Applying the Hasse principle, we see that the orbits are classified by the kernel of the map:

$$H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{G}_2(\alpha, \beta, \gamma)(\mathbb{C})) \rightarrow H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbf{G}_c(\mathbb{C})).$$

For anisotropic groups  $G$  over  $\mathbb{R}$ , the first Galois cohomology yields the set of conjugacy classes of elements of  $G(\mathbb{R})$  of order 1 or 2. The preimage of the trivial conjugacy class in  $\mathbf{G}_c(\mathbb{R})$  is again the trivial conjugacy class, i.e., the distinguished element, of  $\mathbf{G}_2(\alpha, \beta, \gamma)(\mathbb{R})$ . Hence the kernel of the above map is trivial.  $\square$

Now unfolding the expression for  $\Phi_\psi$  yields:

$$\Phi_\psi(g) = \int_{\mathbf{G}_2(\alpha, \beta, \gamma)(\mathbb{A}) \backslash \mathbf{G}_c(\mathbb{A})} \theta'_{\phi_0}(g, g') \oint_{\mathbf{G}_2(\alpha, \beta, \gamma)} f_c(hg') dh dg'.$$

Here,  $\phi_0$  is a fixed  $\phi$  restricting to  $\psi$ . The inner integral is again a period, from which we derive:

**Proposition 5.7.** *If  $f_c$  is not  $\mathbf{G}_2$ -distinguished then  $\Phi$  vanishes. Inversely, if  $f_c$  is  $\mathbf{G}_2$ -distinguished, then  $\Phi$  does not vanish.*

The vanishing of  $\mathbf{G}_2$ -periods clearly implies the vanishing of  $\Phi$  by the equation above. We are left to check the inverse statement; suppose that  $f_c$  is  $\mathbf{G}_2(\alpha, \beta, \gamma)$ -distinguished. Note that:

$$\Phi_\psi(1) = \int_{\mathbf{G}_2(\mathbb{A}) \backslash \mathbf{G}_c(\mathbb{A})} \theta'_\phi(g') \mathcal{P}_{f_c}^{\mathbf{G}_2}(g') dg'.$$

The same analysis as in Section 5, Proposition 4.5 of [20] can now be used to show that the above quantity does not vanish for a suitable choice of  $\theta'$  in the image of  $\Theta'$ .

**5.4. Global theta lift to  $\mathbf{G}_s$ .** We can now study the theta lift from  $\mathbf{G}_c$  to  $\mathbf{G}_s$ . Fix  $\theta$  in the image of  $\Theta$ , and an automorphic form  $f_c \in \mathcal{A}_c = \mathcal{A}(\mathbf{G}_c)$ . The theta lift of  $f_c$  via  $\theta$  is the automorphic form on  $\mathbf{G}_s$  defined by:

$$f_s(g) = \oint_{\mathbf{G}_c} f_c(g') \theta(g, g') dg',$$

where  $g \in \mathbf{G}_s(\mathbb{A})$ , and where we view  $(g, g')$  as an element of  $\mathbf{E}_{8,4}(\mathbb{A})$  via the dual pair embedding:

$$\mathbf{G}_s \times_{\nu} \mathbf{G}_c \hookrightarrow \mathbf{E}_{8,4}.$$

$f_s$  is a cusp form on  $\mathbf{G}_s$  if the constant terms of  $f_s$  along the unipotent radicals of all (standard) maximal parabolic subgroups of  $\mathbf{G}_s$  vanish. The maximal parabolic subgroups of  $\mathbf{G}_s$  are the Heisenberg parabolic  $\mathbf{P}$ , and three parabolic subgroups  $\mathbf{Q}_i = \mathbf{M}_i \mathbf{N}_i$  with  $\mathbf{N}_i$  abelian of dimension 6.

The description of the constant term  $\theta_{\mathbf{H}_E} = c + \theta'$  for  $\theta'$  in the image of  $\Theta'$  shows that if  $f_c$  is orthogonal to the constant functions, then:

$$\begin{aligned} (f_s)_{\mathbf{H}}(g) &= \oint_{\mathbf{H}} \oint_{\mathbf{G}_c} f_c(g') \theta(hg, g') dg' dh, \\ &= \oint_{\mathbf{G}_c} f_c(g') \theta_{\mathbf{H}_E}(g, g') dg', \\ &= \oint_{\mathbf{G}_c} f_c(g') \theta'(g, g') dg'. \end{aligned}$$

To deduce the second line from the first above, note that integration over  $\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A})$  kills all non-constant Fourier coefficients of  $\theta$  except those  $\phi \in \mathbf{F}(\mathbb{Q})$  of the form  $\psi = \begin{pmatrix} 0 & A_+ \\ A_- & 0 \end{pmatrix}$ , with  $A_{\pm}$  having zeroes along their diagonal. But by Lemma 2.7 of [13], and the fact that there are no trace zero, rank one elements of  $\mathbf{J}_3(\mathbb{Q})$ , such  $\psi$  do not exist; the only term to survive is the constant term along all of  $\mathbf{H}_E$ .

Coupled with the results in Proposition 5.7, we have now shown:

**Proposition 5.8.**  *$(f_s)_{\mathbf{H}}$ , as an automorphic form on  $\mathbf{SL}_2^3$ , is the theta lift of  $f_c$  via  $\theta'$ . The constant term of the theta lift  $f_s$  along the Heisenberg parabolic  $\mathbf{H}$  vanishes if  $f_c$  is not  $\mathbf{G}_2$ -distinguished.*

To check whether the lift  $f_s$  is cuspidal, we must determine when  $(f_s)_{\mathbf{N}_i}$  vanishes for the three abelian unipotent radicals  $\mathbf{N}_i$ . We may write  $\mathbf{N}_i = (\mathbf{N}_i \cap \mathbf{H}_E) \oplus \mathbf{N}'_i$ , for a suitable subgroup  $\mathbf{N}'_i$ . Applying this decomposition, we compute these constant terms now:

$$\begin{aligned} (f_s)_{\mathbf{N}_i}(g) &= \oint_{\mathbf{N}_i} \oint_{\mathbf{G}_c} f_c(g') \theta(ng, g') dg' dn, \\ &= \oint_{\mathbf{N}'_i} \oint_{\mathbf{N}_i \cap \mathbf{H}_E} \oint_{\mathbf{G}_c} f_c(g') \theta(n'ng, g') dg' dn' dn, \\ &= \oint_{\mathbf{N}'_i} \oint_{\mathbf{G}_c} f_c(g') \sum_{\psi \in \Omega_i} \theta_{\psi}(n'g, g') dg' dn'. \end{aligned}$$

In the above,  $\Omega_i$  denotes the subset of the orbit  $\Omega$  orthogonal to  $\mathbf{N}_i \cap \mathbf{H}_E$ . Using Gan's classification of elements of  $\Omega$  in Lemma 2.7 of [13], an element of  $\Omega_i$ , for  $i = 1$  looks like:  $\psi = \begin{pmatrix} 0 & A_+ \\ A_- & d \end{pmatrix}$  with  $d \in \mathbb{Q}$  and:

$$A_+ = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & \alpha \\ 0 & \bar{\alpha} & c \end{pmatrix},$$

where  $N(\alpha) = bc$ . The stabilizer of such an element in  $\Omega_i$  is the  $Spin_7$  subgroup  $\mathbf{G}_I(\alpha)$ . From this we see:



**Proposition 5.9.** *If  $f_c$  is not  $\mathbf{G}_I$  (resp.  $\mathbf{G}_{II}, \mathbf{G}_{III}$ ) distinguished, the constant terms  $(f_s)_{N_1}$  (resp.  $(f_s)_{N_2}, (f_s)_{N_3}$ ) vanish.*

If  $f_c$  is not  $\mathbf{G}_2$ -distinguished, then it cannot be  $\mathbf{G}_I$ ,  $\mathbf{G}_{II}$ , or  $\mathbf{G}_{III}$  distinguished, since any of the latter subgroups contains a  $\mathbf{G}_2$  subgroup. Hence we see:

**Theorem 5.10.** *If  $f_c$  is not  $\mathbf{G}_2$ -distinguished, then  $f_s$  is cuspidal.*

Finally, we derive a condition for the theta lift to be non-vanishing. For a generic cube  $c_{ijk}$ , corresponding to a character  $\psi$  of  $\mathbf{H}(\mathbb{A})$ , we compute the Fourier coefficient of  $f_s$  at  $\psi$ . Such a coefficient is given by:

$$(f_s)_\psi(g) = \oint_{\mathbf{G}_c} f_c(g') \sum_{\text{Res}(\phi)=\psi} \theta_\phi(g, g') dg'.$$

Those  $\phi \in \Omega$  that restrict to a generic  $\psi$  have the form  $\phi = x \cdot \begin{pmatrix} 1 & A \\ A^\sharp & \text{Det}(A) \end{pmatrix}$ , for  $x \in \mathbb{Q}^\times$ ,  $A \in \mathbf{J}_3(\mathbb{Q})$ , using the description of  $\Omega$  in Lemma 2.7 of [13] again. If  $A \in \mathbf{J}_3(\mathbb{Q})$  has the form:

$$A = \begin{pmatrix} a & \gamma & \bar{\beta} \\ \bar{\gamma} & b & \alpha \\ \beta & \bar{\alpha} & c \end{pmatrix},$$

then the subgroup of  $\mathbf{G}_c$  stabilizing  $A$  is  $\mathbf{SU}_3(\alpha, \beta, \gamma)$  (generically). An easy computation shows that  $\mathbf{SU}_3(\alpha, \beta, \gamma)$  stabilizes  $A^\sharp$  as well. Thus the stabilizer of  $\phi$  is precisely  $\mathbf{SU}_3(\alpha, \beta, \gamma)$ .

As in Lemma 5.6, we can see that  $\mathbf{G}_c(\mathbb{Q})$  acts transitively on the set of  $\phi$  restricting to  $\psi$  as above. For the orbits are given by the kernel in Galois cohomology of the map:

$$H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbf{SU}_3(\alpha, \beta, \gamma)(\bar{\mathbb{Q}})) \rightarrow H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathbf{G}_c(\bar{\mathbb{Q}})).$$

Once again, both groups are simply connected, and anisotropic over  $\mathbb{R}$ , so the kernel is trivial and there is one  $\mathbf{G}_c(\mathbb{Q})$  orbit.

We continue the computation of the Fourier coefficient by unfolding the integral:

$$(f_s)_\psi(g) = \int_{\mathbf{SU}_3(\alpha, \beta, \gamma)(\mathbb{A}) \backslash \mathbf{G}_c(\mathbb{A})} \theta_{\phi_0}(g, g') \oint_{\mathbf{SU}_3(\alpha, \beta, \gamma)} f_c(hg') dh dg'.$$

Here  $\phi_0$  is a fixed character restricting to  $\psi$ , and again the inner integral is a period:

$$(f_s)_\psi(g) = \int_{\mathbf{SU}_3(\alpha, \beta, \gamma)(\mathbb{A}) \backslash \mathbf{G}_c(\mathbb{A})} \theta_{\phi_0}(g, g') \mathcal{P}^{\mathbf{SU}_3(\alpha, \beta, \gamma)} f_c(g') dg'.$$

The same argument applies to degenerate Fourier coefficients, replacing the group  $\mathbf{SU}_3(\alpha, \beta, \gamma)$ , by a  $\mathbf{G}_2$ ,  $\mathbf{G}_I$ ,  $\mathbf{G}_{II}$ , or  $\mathbf{G}_{III}$  subgroup. In particular, the vanishing of  $\mathbf{SU}_3(\alpha, \beta, \gamma)$  periods implies the vanishing of periods for such larger subgroups.

Hence we have:

**Proposition 5.11.** *If  $f_c$  is not  $\mathbf{SU}_3$ -distinguished, then all Fourier coefficients of the theta-lift  $f_s$  of  $f_c$ , with respect to  $\mathbf{H}/\mathbf{Z}$  vanish.*

The following lemma is adapted from a lemma in Section 8 of [15]:

**Lemma 5.12.** *An automorphic form  $f_s$  on  $\mathbf{G}_s$  vanishes if and only if the constant term  $(f_s)_{\mathbf{Z}}$  vanishes.*

*Proof.* The argument from Lemma 9.1 in [13] applies. Namely, let  $\mathbf{Z}_{1,1,1}$  denote the center of the unipotent radical of the three-step parabolic subgroup  $\mathbf{P}_{1,1,1} \subset \mathbf{G}_s$ . Then  $\mathbf{Z}_{1,1,1}$  is two-dimensional, and contains  $\mathbf{Z}$ . Thus if  $(f_s)_{\mathbf{Z}} = 0$ , then for any character  $\phi$  on  $\mathbf{Z}_{1,1,1}$  trivial on  $\mathbf{Z}$ , we have  $(f_s)_{\phi} = 0$ . But the Levi component  $\mathbf{L}_{1,1,1}$  of  $\mathbf{P}_{1,1,1}$  can be used to take any character  $\phi$  of  $\mathbf{Z}_{1,1,1}$  to a character trivial on  $\mathbf{Z}$ . Thus every Fourier coefficient of  $f_s$  along  $\mathbf{Z}_{1,1,1}$  vanishes, and so  $f_s$  vanishes.  $\square$

We now have:

**Theorem 5.13.** *The theta lift  $f_s$  vanishes if and only if  $f_c$  is not  $\mathbf{SU}_3$ -distinguished.*

*Proof.* If  $f_c$  is not  $\mathbf{SU}_3$ -distinguished, then all Fourier coefficients of  $f_s$  with respect to  $\mathbf{H}/\mathbf{Z}$  vanish. Hence  $(f_s)_{\mathbf{Z}}$  vanishes, and by the previous lemma,  $f_s$  vanishes. The other direction follows from the same analysis as in [20].  $\square$

## 6. EXAMPLES OF MODULAR FORMS

In this section, we discuss the construction of some modular forms on  $\mathbf{G}_c$ , and the resulting modular forms on  $\mathbf{SL}_2^3$  and  $\mathbf{G}_s$ . In particular, a non-zero constant function on  $\mathbf{G}_c$  yields a modular form, whose theta-lift to  $\mathbf{SL}_2^3$  and  $\mathbf{G}_s$  should be identified with Eisenstein series. We consider the significance of the Fourier coefficients of the lift to  $\mathbf{G}_s$ , achieving the arithmetic part of a Siegel-Weil formula.

**6.1. Spherical harmonics and invariants.** A useful way to construct modular forms on  $\mathbf{G}_c$  is through the theory of spherical harmonics, and invariant polynomials. Identify  $\mathbb{O}_c$  with the representation previously denoted  $V_{0,\omega}$ ,  $\omega = (1, 0, 0)$  of  $\mathbf{G}_c(\mathbb{R})$ . Let  $P(n, \mathbb{O}_c)$  denote the space of homogeneous polynomials of degree  $n$  on  $\mathbb{O}_c \simeq \mathbb{R}^8$  with real coefficients. We see that  $P(n, \mathbb{O}_c)$  is a representation of  $\mathbf{Spin}_8 = \mathbf{G}_c(\mathbb{R})$ , identified with  $\text{Sym}^n(V_{0,(1,0,0)})$ . Let  $r^2$  denote the homogeneous polynomial of degree 2 on  $\mathbb{O}_c$ , given by the quadratic norm form. Let  $\Delta$  denote the Laplacian associated to the norm form, normalized so that  $\Delta r^2 = r^2$ .

Define  $H(n, \mathbb{O}_c)$  to be the subspace of  $P(n, \mathbb{O}_c)$  consisting of homogeneous polynomials  $p$  which are harmonic, i.e., satisfy  $\Delta p = 0$ . Then the decomposition of  $P(n, \mathbb{O}_c)$  as a representation of  $\mathbf{Spin}_8$  is well-known:

$$P(n, \mathbb{O}_c) = \bigoplus_{0 \leq m \leq \lfloor n/2 \rfloor} r^{2m} H(n - 2m, \mathbb{O}_c).$$

Each space of harmonic polynomials  $H(n, \mathbb{O}_c)$  is isomorphic as a  $\mathbf{Spin}_8$  representation to  $V_{0,\omega}$  with  $\omega = (n, 0, 0)$ .

From Proposition 3.2 it follows that:

**Proposition 6.1.** *The space of modular forms on  $\mathbf{G}_c$  of level 1, and of weight  $(0, (n, 0, 0))$  can be identified with  $H(n, \mathbb{O}_c)^{\Gamma_c}$ , i.e., with harmonic polynomials of degree  $n$ , invariant under  $\underline{\mathbf{G}}_c(\mathbb{Z})$ .*

In order to study such invariant polynomials for the group  $\Gamma_c$ , we describe a close relationship between  $\Gamma_c$  and the Weyl group  $W_E$  of the  $E_8$  root system. First, we note that the finite group  $\Gamma_c$  is a central extension of  $\underline{\mathbf{G}}_c(\mathbb{F}_2)$  by the abelian group of order 4,  $\nu(\mathbb{Z})$ :

$$1 \rightarrow \nu(\mathbb{Z}) \rightarrow \Gamma_c \rightarrow \underline{\mathbf{G}}_c(\mathbb{F}_2) \rightarrow 1.$$

By our construction of  $\underline{\mathbf{G}}_c$ , we can consider the three images,  $\Gamma_c^I, \Gamma_c^{II}, \Gamma_c^{III}$  of  $\Gamma_c$  in  $\underline{\mathbf{SO}}(\Omega_c, N)$ . Each one of these groups is still a central extension of  $\underline{\mathbf{G}}_c(\mathbb{F}_2)$ , this

time by a group of order 2. In particular, the action of  $\Gamma_c$  on  $\mathbb{O}_c$  factors through the quotient  $\Gamma_c^I$  and:

$$H(n, \mathbb{O}_c)^{\Gamma_c} = H(n, \mathbb{O}_c)^{\Gamma_c^I}.$$

By identifying the  $E_8$  root lattice with Coxeter's octonions (scaling if necessary), the Weyl group  $W_E$  acts on  $\Omega_c$  by the reflection representation  $ref$ . Let  $W_E^+$  denote the kernel of  $det \circ ref$  in  $W_E$ , the subgroup of index 2 acting by proper isometries on  $\Omega_c$ . It is known that  $W_E^+$  is also a central extension:

$$1 \rightarrow \{\pm 1\} \rightarrow W_E^+ \rightarrow \mathbf{G}_c(\mathbb{F}_2) \rightarrow 1.$$

In fact, we have:

**Proposition 6.2.** *There is an isomorphism between  $W_E^+$  and  $\Gamma_c^I$  which intertwines the reflection representation of  $W_E^+$  and the representation of  $\Gamma_c^I$  on  $\Omega_c$ .*

From this, we immediately get:

**Corollary 6.3.** *The space of modular forms on  $\mathbf{G}_c$  of level 1, and of weight  $(0, (n, 0, 0))$  can be identified with  $H(n, \mathbb{O}_c)^{W_E^+}$ .*

The ring of invariants for the full  $E_8$  Weyl group  $W_E$  is generated by invariants of degrees 2, 8, 12, 14, 18, 20, 24, 30. The invariants for the index 2 subgroup  $W_E^+$  consist precisely of the invariants for  $W_E$  and the “skew invariants” as described in [24]. The skew invariants form a free cyclic module over the ring of invariants, generated by a single skew invariant  $Sk_{240}$  of degree 240 (the number of reflections in  $W_E$ ).

One may choose canonical fundamental invariants  $I_2 = r^2, I_8, \dots, I_{30}$ , up to scalar multiple, as described in [22]. These canonical invariants will be harmonic, except for  $I_2$ , which has Laplace eigenvalue 1. The skew invariant  $Sk_{240}$  may be chosen to be harmonic as well, as discussed in [24]. Therefore, we see:

**Theorem 6.4.** *There are canonical, up to scalar multiple, modular forms  $F_d$  on  $\mathbf{G}_c$  of level 1 and of weights  $(0, (d, 0, 0))$  for  $d \in \{8, 12, 14, 18, 20, 24, 30, 240\}$ , associated to the canonical harmonic invariant polynomials,  $I_d$  and the skew invariant polynomial  $Sk_{240}$ . The invariant  $I_2$  of degree 2 corresponds to the trivial representation of  $Spin_8$ , and yields a constant modular form.*

We may apply the results of Corollary 3.13 to see that all of the modular forms  $F_d$  are  $\mathbf{G}_2$ -distinguished. Therefore, the theta-lifts of  $F_d$  to  $\mathbf{SL}_2^3$  and to  $\mathbf{G}_s$  do not vanish. In order to understand these  $F_d$ , and their theta-lifts, one must first be able to explicitly write down the canonical polynomial invariants  $I_d$ , preferably in a way that exploits the octonionic structure of the  $E_8$  root lattice. We leave this study to a future paper.

**6.2. Lifting the trivial modular form to  $SL_2^3$ .** Though the constant modular form on  $\mathbf{G}_c$  is uninteresting by itself, its theta-lifts are worthy of study, especially considering the exceptional Siegel-Weil formula of Gan [14]. We describe the theta-lifts to  $\mathbf{SL}_2^3$  and  $\mathbf{G}_s$  here, focusing on connections to Eisenstein series, and a description of Fourier coefficients.

The lifting of the trivial modular form on  $\mathbf{G}_c$  to  $\mathbf{SL}_2$  is particularly simple, given Kim's thorough description of a theta function on  $\mathbf{E}_{7,3}$ , as well as some

more general work of Gross-Elkies in [11]. In particular, we can work classically throughout, beginning with the theta function on the exceptional tube domain:

$$\theta'(Z) = 1 + 240 \sum_{A \geq 0, rk(A)=1} \left( \sum_{d|c(A)} d^3 \right) e^{2\pi i \langle A, Z \rangle}.$$

Some explanation is necessary for the above formula. We view  $\theta'$  as a holomorphic function on the exceptional tube domain:

$$\mathcal{D} = \{Z = X + iY : X \in \mathbf{J}_3(\mathbb{R}), Y \in \mathbf{J}_3(\mathbb{R})_+\}.$$

The summation is over elements  $A \in \mathbf{J}_3(\mathbb{Z})$ , of rank 1, which are positive semi-definite. If  $A \in \mathbf{J}_3(\mathbb{Z})$ , the integer  $c(A)$  refers to the largest positive integer dividing  $A$ . The embedding of  $\mathbf{G}_c \times_\nu \mathbf{SL}_2^3$  in  $\mathbf{E}_{7,3}$ , and the action of  $\mathbf{E}_{7,3}(\mathbb{R})$  on the exceptional tube domain, allows us to define the theta-lift of the constant modular form on  $\mathbf{G}_c$  as:

$$\Phi(z_1, z_2, z_3) = \int_{\mathbf{G}_c(\mathbb{R})} \theta' \left( g \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} \right) dg.$$

For  $Z$  a diagonal element of the exceptional tube domain, and  $g \in \mathbf{G}_c(\mathbb{R})$  as in the above formula,  $gZ = Z$ . Therefore, we see that:

$$e^{2\pi i \langle A, gZ \rangle} = e^{2\pi i \langle A, Z \rangle}.$$

Hence Kim's exceptional form is invariant under translation by  $\mathbf{G}_c(\mathbb{R})$ . It follows immediately that the modular form  $\Phi$  on  $\mathbf{SL}_2^3$  is given by:

$$\Phi(z_1, z_2, z_3) = \sum_{a_1, a_2, a_3 \in \mathbb{N}} \rho(a) e^{2\pi i (a_1 z_1 + a_2 z_2 + a_3 z_3)},$$

where the coefficients  $\rho$  are given by:

$$\rho(a) = 240 \sum_{rk(A)=1, diag(A)=a} \left( \sum_{d|c(A)} d^3 \right).$$

From Proposition 4.1 of [11], or noticing the connection between these coefficients and the theta function for the  $E_8$  root lattice, we get:

**Theorem 6.5.** *The theta-lift of the constant function on  $\mathbf{G}_c$  via Kim's exceptional form on  $\mathbf{E}_{7,3}$  to a modular form on  $\mathbf{SL}_2^3$  is the product of three Eisenstein series of weight 4 for  $\mathbf{SL}_2$ :*

$$\Phi(z_1, z_2, z_3) = E_4(z_1)E_4(z_2)E_4(z_3).$$

This may be seen as an analogue of the Siegel-Weil formula for the dual pair  $\mathbf{G}_c \times_\nu \mathbf{SL}_2^3$  in  $\mathbf{E}_{7,3}$ , though it does not contain significant new arithmetic information.

**6.3. Lifting the trivial modular form to  $G_s$ .** Far more interesting to us is the lifting of the trivial modular form to a modular form on  $\mathbf{G}_s$ . For this, we fix  $t_f = \bigotimes t_p$ , the product of the normalized spherical vectors of the minimal representation of  $\mathbf{E}_{8,4}(\mathbb{Q}_p)$  for all (finite) primes  $p$ . By the archimedean theta correspondence of Loke, which we discussed in Theorem 4.1, we have a (unique up to scaling) embedding  $\iota$  of the quaternionic discrete series representation  $\pi_{10}$

of  $\mathbf{G}_s(\mathbb{R})$ , paired with the trivial representation of  $\mathbf{G}_c(\mathbb{R})$  into the local minimal representation  $\Pi_\infty$  of  $\mathbf{E}_{8,4}(\mathbb{R})$ .

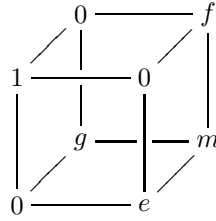
For any vector  $v$  of the quaternionic discrete series  $\pi_{10}$ , embedded as a vector  $\iota(v)$  in  $\Pi_\infty$ , we have an automorphic form  $\Theta(t_f \otimes \iota(v))$  on  $\mathbf{E}_{8,4}(\mathbb{A})$ . Restricting from  $\mathbf{E}_{8,4}(\mathbb{A})$  to  $\mathbf{G}_s(\mathbb{A})$  yields an automorphic form  $\Psi_v$  on  $\mathbf{G}_s(\mathbb{A})$ , and the map  $v \mapsto \Psi_v$  is a modular form of weight 10 and level 1, in the sense of Definition 2.3.

Hereafter, we denote by  $\Psi$  the modular form obtained by lifting the trivial modular form on  $\mathbf{G}_c$  as above. Since  $\Psi$  has level one, there are well-defined Fourier coefficients  $a_c$  for  $\Psi$ , indexed by 2 by 2 by 2 cubes  $c$ . For simplicity, we consider only those coefficients corresponding to projective, non-degenerate cubes; in this case, the  $SL_2(\mathbb{Z})^3$ -invariance of the Fourier coefficients allows us to consider only cubes in normal form as well.

Following the work in Section 11 of [14], we have, after suitably normalizing all coefficients:

**Proposition 6.6.** *If  $c$  is a projective non-degenerate cube, then the Fourier coefficient  $a_c$  is equal to the number of elements  $\omega \in \Omega \cap \mathbf{F}(\mathbb{Z})$  which restrict to the cube  $c$ .*

The counting problem in this proposition is tractable, when  $c$  is in normal form. We let  $c$  be the cube:



An element  $\omega \in \Omega \cap \mathbf{F}(\mathbb{Z})$  restricting to  $c$  is then a matrix  $\begin{pmatrix} 1 & A \\ A^\sharp & \text{Det}(A) \end{pmatrix}$ , where the diagonal entries of  $A^\sharp$  are  $e, f, g$ ,  $\text{Det}(A) = m$ , and  $A$  is given by:

$$A = \begin{pmatrix} 0 & \gamma & \bar{\beta} \\ \bar{\gamma} & 0 & \alpha \\ \beta & \bar{\alpha} & 0 \end{pmatrix}.$$

Hence we see that:

**Proposition 6.7.** *The number of elements  $\omega \in \Omega \cap \mathbf{F}(\mathbb{Z})$  restricting to  $c$  (and hence the Fourier coefficient  $a_c$ ) is equal to the number of triples  $(\alpha, \beta, \gamma) \in \Omega_c^3$  such that  $N(\alpha) = -e$ ,  $N(\beta) = -f$ ,  $N(\gamma) = -g$ , and  $\text{Tr}(\alpha\beta\gamma) = m$ .*

**6.4. An embedding problem.** The previous proposition gives some interpretation of the Fourier coefficients of the theta lift of the constant function on  $\mathbf{G}_c$ . However, in analogy to classical Siegel-Weil formulas, and the exceptional Siegel-Weil formula of Gan [14], we look for a more arithmetically interesting interpretation. For this, we introduce the following algebraic object:

**Definition 6.8.** A  $QT$ -structure (over  $\mathbb{Z}$ ) of rank  $n$  consists of a free  $\mathbb{Z}$ -module  $\Lambda$  of rank  $n$ , three integer-valued quadratic forms  $Q_1, Q_2, Q_3$  on  $\Lambda$ , and a trilinear form  $T: \Lambda \otimes \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ .

We have already seen one  $QT$ -structure, namely Coxeter's integral octonions  $\Omega_c$ , endowed with the trilinear form  $Tr(\alpha\beta\gamma)$ , letting all three quadratic forms  $Q_i$  be the norm form. We can deduce from results of Bhargava in [4], that every 2 by 2 by 2 cube also yields a  $QT$ -structure as well. We describe this construction here:

Suppose that the cube  $c$  is in normal form as before, with discriminant  $D \neq 0$ , and let  $R(D)$  denote the quadratic ring of discriminant  $D$ . Associated to the cube  $c$ , we get three invertible oriented ideal classes  $I_1, I_2, I_3$ . Let  $\Lambda$  denote the free  $\mathbb{Z}$ -module of rank 2, with basis  $\lambda, \mu$ . By results in the Appendix of [4], there exist  $\mathbb{Z}$ -module isomorphisms from  $\Lambda$  to  $I_1, I_2, I_3$  (choosing appropriate representatives for these ideal classes), such that the quadratic norm form on  $R(D) \otimes \mathbb{Q}$ , applied to the images of  $x\lambda + y\mu$  in  $I_1, I_2, I_3$  yields the following binary quadratic forms:

$$\begin{aligned} Q_1 &= -ex^2 + mxy + fgy^2, \\ Q_2 &= -fx^2 + mxy + egy^2, \\ Q_3 &= -gx^2 + mxy + efy^2. \end{aligned}$$

In this way, we get three quadratic forms  $Q_1, Q_2, Q_3$  on the lattice  $\Lambda$ , by looking at the norm form on  $I_1, I_2, I_3$ . In order to get a trilinear form, we look at the trilinear map which multiplies elements of  $I_1, I_2, I_3$  (viewing them as elements of  $R(D) \otimes \mathbb{Q}$ ), and applies the trace map from  $R(D)$  to  $\mathbb{Z}$ . With the same basis  $x, y$  for  $\mathbb{Z}^2$  as above, we may compute the values of the trilinear form; for example  $T(\lambda, \lambda, \lambda) = m$  and  $T(\mu, \mu, \mu) = m^2 + 2efg$ . In any case, we see that a 2 by 2 by 2 cube, in normal form, yields a  $QT$ -structure  $(\Lambda, Q, T)$  of rank 2. Moreover, with respect to a well-chosen basis  $\lambda, \mu$  for this structure, we have  $Q_1(\lambda) = -e$ ,  $Q_2(\lambda) = -f$ ,  $Q_3(\lambda) = -g$ , and  $T(\lambda, \lambda, \lambda) = m$ . With a few tedious arithmetic computations, and Proposition 6.7, we get:

**Theorem 6.9.** *The Fourier coefficients  $a_c$  associated to any projective non-degenerate cube  $c$  in normal form count the number of embeddings of the  $QT$ -structure associated to  $c$  into the  $QT$ -structure coming from Coxeter's integral octonions.*

This can be seen as the arithmetic part of a Siegel-Weil formula for the dual pair  $(\mathbf{G}_c, \mathbf{G}_s)$  in  $\mathbf{E}_{8,4}$ . It would be interesting if one could identify the modular form  $\Psi$  with an Eisenstein series on  $\mathbf{G}_s$ , and use this to obtain other formulae for the  $a_c$ , especially relating to values of L-functions.

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